4

Gamma and Beta Function
(Some Special Integrals)

The Gamma function and Beta function are the integral special functions and are considered as generalisation of the factorial function which involves integral with different limits. The Gamma function is defined as the single variable function whereas Beta function is defined as the two variable function. These two functions are however connected to each other and many complex integral can be reduced and evaluated by using these functions. In physics these functions are popularly used in quantum electrodynamics, quantum chromodynamics (renormalisation group equation), cosmology etc. These functions are also used as inbuilt functions in scientific softwares like mathematica, matlab etc. and helps in numerical computations.

1. GAMMA FUNCTION

The Gamma function which is also known as Euler's integral of the second kind was first introduced by Euler in his work to generalize the factorial of non-integer values. Gamma function can be defined as (out of three different convenient definitions),

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, \text{ where } n > 0 \qquad ...(1.1)$$

This function depends only upon n not on x.

So, $$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1}\, dx$$

Thus, $$\Gamma(4) = \int_0^{\infty} e^{-x} x^{4-1}\, dx \text{ and so on.}$$

1.1 PROPERTIES OF GAMMA (Γ) FUNCTIONS

Some interesting properties of Gamma function are

(*a*) $\Gamma(1) = 1$

(*b*) $\Gamma(n+1) = n\Gamma(n) = n!$

(*c*) $\Gamma(n) = z^n \int_0^{\infty} e^{-zx} x^{n-1} dx$, where $n, z > 0$

(*d*) $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$

(*e*) $\Gamma(n+1) = \int_0^{\infty} e^{-y^{\frac{1}{n}}} dy$ and so on.

Proof

Let us prove the properties one by one

(*a*) To prove $\quad \Gamma(1) = 1$

We know, $\quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad$ [From equation (1.1)]

$\therefore \quad \Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx \quad$ [Putting $n = 1$]

$$= \int_0^\infty e^{-x}\, dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -1\,(0-1) = 1 \qquad [\because e^{-\alpha} = 0]$$

$\therefore \quad \mathbf{\Gamma(1) = 1}$

(*b*) To prove $\quad \Gamma(n+1) = n\Gamma(n) = n!$

We know $\quad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad$ [From equation (1.1)]

Now putting $n = n + 1$, we get

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1} dx$$

$$= \int_0^\infty e^{-x} x^n dx \qquad ...(1.2)$$

Integrating by parts on the *r.h.s.*

$$\Gamma(n+1) = x^n \int_0^\infty e^{-x} dx - \int_0^\infty \left\{\frac{d}{dx}(x^n) \int e^{-x}\right\} dx$$

$$= \left[x^n \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} dx$$

$$= -(0-0) + n \int_0^\infty x^{n-1} e^{-x} dx \qquad ...(1.3)$$

[using equation (1.1)]

$\therefore \quad \mathbf{\Gamma(n+1) = n\Gamma(n)}$

Using the same property, we write $\Gamma(n) = (n-1)\,\Gamma(n-1)$ (Putting $n = n - 1$, in eq. 1.3)

$$\therefore \quad \Gamma(n+1) = n\,(n-1)\,\Gamma(n-1)$$

$$= n\,(n-1)\,(n-2)\,\Gamma(n-2)$$

$$= n\,(n-1)\,(n-2)\,(n-3)\,\Gamma(n-3)$$

$$= n\,(n-1)\,(n-2)\,(n-3)\ ...3.2.1\ \Gamma.\ (1)$$

$$\mathbf{\Gamma(n+1) = n!} \qquad ...(1.4)\ [\because \Gamma(1) = 1]$$

Thus, $\quad \Gamma(2) = 1! = 1$

$\Gamma(3) = 2! = 2$

$\Gamma(4) = 3! = 6$ and so on.

From equation (1.3), we get $\quad \Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

If $n = 0$, then $\Gamma(0) = \dfrac{\Gamma(1)}{0} = \dfrac{1}{0} = \infty$

And if $n = -1$,

Using $\Gamma(n) = (n-1)\,\Gamma(n-1)$ one gets

$$\Gamma(-1) = (-1-1)\,\Gamma(-1-1)$$

$$= -2\,\Gamma(-2)$$

Similarly $\Gamma(-2) = -3\Gamma(-3)$ so that

$$\Gamma(-1) = (-2)(-3)\,\Gamma(-3)$$

Proceeding in a same way one will get

$$\Gamma(-1) = (-1)^m\, m\,!\,\Gamma(-m)$$

where $m \to \infty$ hence gamma function for any negative integer is not defined.

then $\Gamma(-1) = \infty$

So this definition is not **valid for zero or negative integers.** But we can find the value of Gamma function for negative non integer value of *n*.

Now putting $n = -\frac{1}{2}$ in equation (1.3),

$$\Gamma\left(-\frac{1}{2}\right) = -2\Gamma\left(\frac{1}{2}\right)$$

where $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{1-\frac{1}{2}}\, e^{-x}\, dx$

$$= \int_0^\infty x^{\frac{1}{2}}\, e^{-x}\, dx$$

that will be calculated in the example 1.1 (*a*)

Likewise $\Gamma\left(-\frac{3}{2}\right) = \dfrac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}$

$$= -\frac{2}{3}\,\Gamma\left(-\frac{1}{2}\right)$$

$$= \left(-\frac{2}{3}\right)(-2)\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{4}{3}\,\Gamma\left(\frac{1}{2}\right)$$

Repetition of the above identity allow us to define the Gamma function on the whole real axis except on the negative integers as shown in the fig. 1.1.

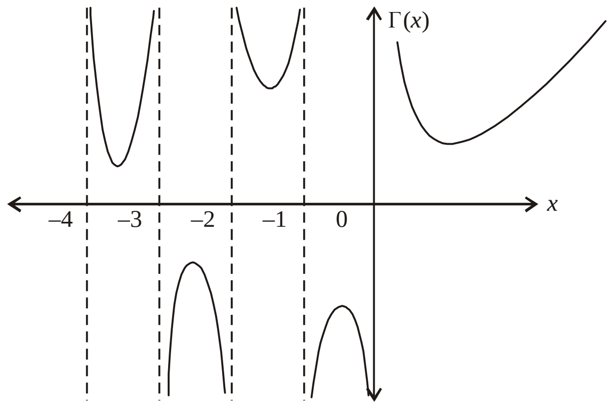

Fig. 1.1. Graphical representation of Gamma Function

(c) To prove $$\boldsymbol{\Gamma(n) = z^n \int_0^\infty e^{-zx} x^{n-1} dx}$$

By definition of the gamma function, we get

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad \text{[From equation (1.1)]}$$

Let $x = zy$

or $dx = z\,dy$

Here limit will not be changed, so replacing the above values

$$\Gamma(n) = \int_0^\infty e^{-zy} (zy)^{n-1} z\,dy$$

$$= \int_0^\infty e^{-zy} z^{n-1} y^{n-1} z\,dy$$

$$= z^n \int_0^\infty e^{-zy} y^{n-1} dy \qquad ...(1.5)$$

For a definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy \qquad ...(1.6)$$

Therefore from equations (1.5) and (1.6), one gets

$$\boldsymbol{\Gamma(n) = z^n \int_0^\infty e^{-zx} x^{n-1} dx} \qquad ...(1.7)$$

(d) To prove $$\boldsymbol{\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy}$$

We know that

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad ...(1.8)$$

Let $x = \log \dfrac{1}{y}$

or $e^x = \dfrac{1}{y}$

or $y = e^{-x}$ or $dy = -e^{-x}\, dx$

Now, when $x = 0$, $y = 1$

and when $x = \infty$, $y = 0$

So, changing the variables in equation (1.8),

we get, $$\Gamma(n) = \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} (-dy)$$

$\therefore$ $\boldsymbol{\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy}$ (negative sign is omitted by changing the limit)

[**Note :** This is the first formula derived by Euler for Gamma Function]

(*e*) To prove $$\boldsymbol{\Gamma(n+1) = \int_0^\infty e^{-y^{\frac{1}{n}}} dy}$$

We know that $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad \text{[From equation (1.1)]}$$

Let $$x = y^{\frac{1}{n}}$$

or $$x^n = y$$

or $$nx^{n-1}\, dx = dy$$

or $$x^{n-1}\, dx = \frac{dy}{n}$$

Here limits will not be changed as when $x = 0$, $y = x^4 = 0$ and when $x = \infty$, $y = \infty$.

$\therefore$ $$\Gamma(n) = \int_0^\infty e^{-y^{\frac{1}{n}}} \frac{dy}{n} \qquad \text{[changing the variable in equation (1.1)]}$$

$$= \frac{1}{n} \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

or $$n\,\Gamma(n) = \int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

or $$\boldsymbol{\Gamma(n+1) = \int_0^\infty e^{-y^{\frac{1}{n}}} dy} \qquad \text{(Using equation 1.3)} \qquad ...(1.9)$$

Example 1.1. Find the values of

(*i*) $\Gamma\left(\frac{1}{2}\right)$, **(*ii*)** $\Gamma\left(\frac{3}{2}\right)$, **(*iii*)** $\Gamma\left(\frac{5}{2}\right)$, **(*iv*)** $\Gamma\left(-\frac{1}{2}\right)$, **(*v*)** $\Gamma\left(-\frac{3}{2}\right)$

Solution : (*i*) We know that, $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

For $n = \frac{1}{2},\ \Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$ ...(*i*)

Let us put $x = u^2 \quad \therefore \quad x^{-\frac{1}{2}} = u^{-1}$

$\therefore \quad dx = 2udu$

When $x = 0,\ u = 0$

$x = \infty,\ u = \infty$

From $(i) \Rightarrow \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2} u^{-1} 2udu$

$$= 2\int_0^\infty e^{-u^2} du \qquad ...(ii)$$

Along x and y axes equation (*ii*) becomes, (replacing $u = x$ and $u = y$)

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx \qquad ...(iii)$$

And $$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy \qquad ...(iv)$$

Let $x = r\cos\theta,\ y = r\sin\theta$

$\therefore \quad x^2 + y^2 = r^2,\ dxdy = r\,dr\,d\theta$

Here the coordinates are used in 2D cylindrical co-ordinates in which limit of r is 0 to ∞ and θ is 0 to $\pi/2$.

Multiplying equations (*iii*) with (*iv*),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

$$= 2\int_0^{\frac{\pi}{2}} \left\{ \int_0^\infty 2re^{-r^2}\ dr \right\} d\theta$$

Again putting $r^2 = v$

$\therefore \quad 2rdr = dv$

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\int_0^{\frac{\pi}{2}} \left\{ \int_0^\infty e^{-v} dv \right\} d\theta$$

$$= 2\Gamma(1) \int_0^{\frac{\pi}{2}} d\theta \qquad \left[\because \Gamma(1) = \int_0^\infty e^{-v} dv\right]$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta \qquad [\because \Gamma(1) = 1]$$

$$= 2[\theta]_0^{\frac{\pi}{2}}$$

$$= \pi$$

$$\therefore \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(*ii*) We get from the properties of Gamma function

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1) \qquad \text{[From equation (1.3)]}$$

or

$$\Gamma(n+1) = n\Gamma(n)$$

$$\therefore \quad \Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right)$$

$$= \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{1}{2}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

(*iii*) We write

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{3}{2}\Gamma\left(\frac{3}{2}\right) \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

(*iv*) We know that,

$$\Gamma(n+1) = n\,\Gamma(n) \qquad \text{[From equation (1.3)]}$$

Putting $n = -\frac{1}{2}$ in the above equation

$$\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

or $$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

or $$\sqrt{\pi} = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

or $$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

(*v*) Again we know that

$$\Gamma(n+1) = n\,\Gamma(n)$$

Putting $n = -\frac{3}{2}$ in this equation

$$\Gamma\left(-\frac{3}{2}+1\right) = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right)$$

or $$\Gamma\left(-\frac{1}{2}\right) = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right)$$

or $$-2\sqrt{\pi} = -\frac{3}{2}\Gamma\left(-\frac{3}{2}\right) \qquad \left[\because \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$

or $$\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

<u>Example 1.2.</u> Evaluate the integrals by using the property of Γ function.

(*a*) $\int_0^\infty x^4 e^{-x} dx$ **(*b*)** $\int_0^\infty x^8 e^{-4x} dx$ **(*c*)** $\int_0^\infty x^{-t} t^{\frac{7}{2}} dt$

(*d*) $\int_0^\infty \sqrt{x} e^{-3\sqrt{x}} dx$ **(*e*)** $\int_0^\infty e^{-4x} x^{\frac{5}{2}}\, dx$

[**Note :** In evaluating the integrals, we need to check the exponential part. If it is not in e^{-x} form, then we may move for variable replacement]

Solution :

(*a*) We know that

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$\therefore \quad \int_0^\infty x^4 e^{-x} dx = \Gamma(5) \qquad \text{(Putting } n = 5\text{)}$$

Again $$\Gamma(n+1) = n\,!$$

$$\Gamma(5) = 4! = 24$$

$$\therefore \quad \int_0^\infty x^4 e^{-x} dx = 24$$

(***b***) We know that

$$\Gamma(n) = z^n \int_0^\infty e^{-zx} x^{n-1} dx \qquad \text{[From equation (1.7)]}$$

or $$\int_0^\infty e^{-zx} x^{n-1} dx = \frac{\Gamma(n)}{z^n}$$

Here $n - 1 = 8$ or $n = 9$

and $z = 4$

$\therefore$ $$\int_0^\infty x^8 e^{-4x} dx = \frac{\Gamma(9)}{4^9}$$

$$= \frac{8!}{4^9} \qquad [\because \Gamma(n+1) = n!]$$

$$= \frac{40320}{262144} = 0.153$$

(***c***) We know that

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt$$

Here $n - 1 = \frac{7}{2}$ or $n = 1 + \frac{7}{2} = \frac{9}{2}$

$\therefore$ $$\int_0^\infty e^{-t} t^{\frac{7}{2}} dt = \Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2} + 1\right)$$

$$= \frac{7}{2} \Gamma\left(\frac{7}{2}\right) \qquad [\because \Gamma(n+1) = n\,\Gamma(n)]$$

$$= \frac{7}{2} \cdot \frac{5}{2} \cdot \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{105}{16} \sqrt{\pi}$$

(***d***) We know that

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Let $$I = \int_0^\infty \sqrt{x} e^{-3\sqrt{x}} dx$$

Again let, $$3\sqrt{x} = y$$

or $$3 \frac{1}{2} x^{-\frac{1}{2}} dx = dy$$

or $$dx = \frac{2}{3}\sqrt{x}\,dy = \frac{2}{3}\frac{y}{3}dy = \frac{2}{9}y\,dy$$

Here no change in limit will occur.

$\therefore$ $$I = \int_0^\infty \frac{y}{3}e^{-y}\frac{2}{9}y\,dy$$

$$= \frac{2}{27}\int_0^\infty y^2 e^{-y}dy$$

$$= \frac{2}{27}\int_0^\infty y^{3-1}e^{-y}dy$$

$$= \frac{2}{27}\Gamma(3)$$

$$= \frac{2}{27}\ 2! \qquad [\because\ \Gamma(3) = 2!]$$

$$= \frac{4}{27}$$

(*e*) We know that

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

Let $$I = \int_0^\infty e^{-4x}x^{\frac{5}{2}}dx$$

Let $4x = y$

or $4dx = dy$

or $dx = \frac{dy}{4}$

There is no change of limit here.

$\therefore$ $$I = \int_0^\infty e^{-y}\left(\frac{y}{4}\right)^{\frac{5}{2}}\frac{dy}{4}$$

$$= \int_0^\infty e^{-y}y^{\frac{5}{2}}\frac{dy}{4\times(4)^{\frac{5}{2}}}$$

$$= \frac{1}{4\times4\times4\times\sqrt{4}}\int_0^\infty e^{-y}y^{\frac{7}{2}-1}dy$$

$$= \frac{1}{128}\int_0^\infty e^{-y}y^{\frac{7}{2}-1}dy$$

$$= \frac{1}{128}\Gamma\left(\frac{7}{2}\right) = \frac{1}{128}\times\Gamma\left(\frac{5}{2}+1\right) = \frac{1}{128}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15}{128\times 2\times 2\times 2}\sqrt{\pi}$$

$$= \frac{15}{1024}.\sqrt{\pi}$$

Example 1.3. **Show that** $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$**.**

Solution : By definition of Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \qquad \text{[From equation (1.1)]}$$

Let $I = \int_0^\infty e^{-x^2} dx$

Consider $x^2 = y$

or $2x\, dx = dy$

or $dx = \frac{dy}{2\sqrt{y}}$

There will not be any change in limit here.

$$\therefore \quad I = \int_0^\infty e^{-y}\frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{-\frac{1}{2}}dy$$

$$= \frac{1}{2}\int_0^\infty e^{-y}\, y^{\frac{1}{2}-1}dy$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}$$

Example 1.4. **Show that** $\int_0^\infty x^n e^{-kx^2}\, dx = \frac{1}{2k^{n+1}}\Gamma\left(\frac{n+1}{2}\right)$**.**

Solution : By definition of the gamma function, we get

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

Let $\quad I = \int_0^\infty x^n e^{-k^2x^2}\,dx$

Put $\quad k^2x^2 = y$

or $\quad k^2\, 2x\, dx = dy$

$$dx = \frac{dy}{2k^2x} = \frac{dy}{2k^2\sqrt{\frac{y}{k^2}}} = \frac{dy}{2k\sqrt{y}}$$

Here no change in limit will occur,

$$\therefore \quad I = \int_0^\infty \left[\left(\frac{y}{k^2}\right)^{\frac{1}{2}}\right]^n e^{-y}\,\frac{dy}{2k\sqrt{y}}$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{\frac{n}{2}}}{k^n} e^{-y}\, y^{-\frac{1}{2}}\,\frac{dy}{k}$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{\frac{n}{2}-\frac{1}{2}}e^{-y}}{k^{n+1}}\,dy$$

$$= \frac{1}{2k^{n+1}}\int_0^\infty y^{\frac{n-1}{2}}\,e^{-y}\,dy$$

$$= \frac{1}{2k^{n+1}}\int_0^\infty y^{\frac{n+1}{2}-1}\,e^{-y}\,dy$$

$$\therefore \quad I = \frac{1}{2k^{n+1}}\Gamma\left(\frac{n+1}{2}\right) \qquad \text{[By definition of Gamma function]}$$

<u>Example 1.5.</u> Prove that $\Gamma\left(\frac{1}{2}\right)\Gamma(2n) = 2^{2n-1}\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)$

Solution : We have, $\Gamma\left(n+\frac{1}{2}\right)$

$$= \Gamma\left(n-\frac{1}{2}+1\right)$$

$$= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\Gamma\left(\frac{2n-5}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\ldots\left\{\frac{2n-(2n-1)}{2}\right\}\Gamma\left(\frac{2n-2n+1}{2}\right)$$

$$= \frac{(2n-1)\,(2n-3)\,(2n-5)\ldots.1\Gamma\left(\frac{1}{2}\right)}{2^n}$$

Multiply and divide the above relation with $2n\,(2n-2)\,(2n-4)$ ...... 6.4.2.1, one gets

$$= \frac{(2n)!\,\Gamma\left(\frac{1}{2}\right)}{2^n 2n(2n-2)\,(2n-4)\ldots 6.4.2.1}$$

$$= \frac{(2n)\,!\Gamma\left(\frac{1}{2}\right)}{2^{2n}\;n!}$$

$$= \frac{2n\Gamma\,(2n)\,\Gamma\left(\frac{1}{2}\right)}{2^{2n}\;n\Gamma(n)} = \frac{\Gamma(2n)\,\Gamma\left(\frac{1}{2}\right)}{2^{2n-1}\,\Gamma(n)}$$

Thus $$\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma\,(2n)\,\Gamma\left(\frac{1}{2}\right)}{2^{2n-1}\Gamma(n)}$$

$\therefore$ $$\Gamma\left(\frac{1}{2}\right)\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)$$

Example 1.6. **Prove that $\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$**

Solution : The given integral is

$$\text{I} = \int_0^1 x^m (\log x)^n\,dx$$

Putting $x = e^{-y} \Rightarrow e^y = \frac{1}{x}$; $dx = -e^{-y}\,dy$

Now, when $x = 0,\; e^y = \frac{1}{0}$ or $e^y = \infty$ or $y = \infty$

and when $x = 1,\; e^y = \frac{1}{1}$ or $e^y = 1$ or $e^y = e^0 \Rightarrow y = 0$

$$\therefore \qquad I = \int_{\infty}^{0} e^{-my} (-y)^n (-e^{-y})\, dy$$

$$= (-1)^n \int_0^{\infty} e^{-my} y^n e^{-y} dy$$

(limit is changed by using the negative sign)

$$= (-1)^n \int_0^{\infty} y^n\, e^{-(m+1)y}\, dy$$

$$= (-1)^n \int_0^{\infty} e^{-u} \left(\frac{u}{m+1}\right)^n \frac{du}{m+1} \qquad \text{[Putting } (m+1)\,y = u]$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^{\infty} e^{-u} u^n\, du$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1) = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

$$\therefore \qquad \int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Example 1.7. **Show that 2.4.6. .... 2n = $2^n\Gamma$ (n + 1)**

Solution :

$$\text{LHS} = 2.4.6.... 2n$$
$$= 2^n (1.2.3 \ldots n)$$
$$= 2^n n!$$
$$= 2^n \Gamma(n+1) = \text{RHS}$$

Example 1.8. **Show that 1.3.5 .... (2n – 1) = $\dfrac{2^{1-n}\,\Gamma(2n)}{\Gamma(n)}$**

Solution :

$$\text{LHS} = 1.3.5. \ldots (2n-1)$$

$$= \frac{1.2.3.4.5....(2n-1)\ 2n}{2.4.6....2n}$$

$$= \frac{(2n)!}{2^n n!}$$

$$= \frac{2n\Gamma(2n)}{2^n n\Gamma(n)} = \frac{2^{1-n}\Gamma(2n)}{\Gamma(n)} = \text{RHS}$$

Example 1.9. **Evaluate $\int_0^{\infty} e^{-ax} x^{m-1} \sin bx\, dx$ in terms of Gamma Function.**

Solution : The integral,

$$I = \int_0^{\infty} e^{-ax} x^{m-1} \sin bx\, dx$$

$$= \int_0^\infty e^{-ax} x^{m-1} \left(\frac{e^{ibx} - e^{-ibx}}{2i} \right) dx$$

(Putting the value of sin *bx* in terms of exponential function)

$$= \frac{1}{2i} \left[\int_0^\infty e^{-ax} \; x^{m-1} e^{ibx} \; dx - \int_0^\infty e^{-ax} \; x^{m-1} \; e^{-ibx} \; dx \right]$$

$$= \frac{1}{2i} \left[\int_0^\infty e^{-(a-ib)x} \; x^{m-1} \; dx - \int_0^\infty e^{-(a+ib)x} \; x^{m-1} \; dx \right]$$

Let us consider $(a - ib)\, x = u$; $(a - ib)\, dx = du$

and $(a + ib)\, x = v$; $(a + ib)\, dx = dv$

Now the integral becomes

$$\text{I} = \frac{1}{2i} \left[\int_0^\infty \frac{e^{-u} u^{m-1} du}{(a-ib)^m} - \int_0^\infty \frac{e^{-v} v^{m-1} dv}{(a+ib)^m} \right]$$

$$= \frac{1}{2i} \left[\frac{\Gamma(m)}{(a-ib)^m} - \frac{\Gamma(m)}{(a+ib)^m} \right]$$

$$= \frac{\Gamma\,(m)}{2i} \; [(a - ib)^{-m} - (a + ib)^{-m}]$$

Again let $a = r \cos \theta$ and $b = r \sin \theta$

So $\quad a^2 + b^2 = 1$

or $\quad r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \left(\frac{b}{a} \right)$

Putting the value of a and b, we get

$$\text{I} = \frac{\Gamma(m)}{2i} \; [r^{-m} (\cos m\theta + i \sin m\,\theta - \cos m\,\theta + i \sin m\,\theta)]$$

or
$$\text{I} = \frac{\Gamma(m)}{2i} \; r^{-m}\, 2i \sin m\theta$$

or
$$\text{I} = \frac{\Gamma(m)}{r^m} \sin m\theta$$

Finally,
$$\int_0^\infty e^{-ax} x^{m-1} \sin bx \; dx = \frac{\Gamma(m)}{r^m} \sin m\theta$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \left(\frac{b}{a} \right)$

Example 1.10. **Prove that** $\int_0^\infty \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$

Solution : Let

$$I = \int_0^\infty \frac{x^a}{a^x} dx$$

Considering log on both sides of a^x, one gets

$$\log a^x = x \log a$$

Considering the exponent of the above eqn. one gets

$$e^{\log a^x} = e^{x \log a}$$

or

$$a^x = e^{x \log a}$$

$$= \int_0^\infty \frac{x^a}{e^{x \log a}} dx$$

$$= \int_0^\infty x^a e^{-x \log a} dx$$

Let

$$x \log a = z$$

or

$$dx = \frac{dz}{\log a}$$

$\therefore$

$$I = \int_0^\infty \left(\frac{z}{\log a}\right)^a e^{-z} \frac{dz}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty z^a e^{-z} dz$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

Example 1.11. **Evaluate** $\int_0^{\frac{\pi}{2}} (\tan^3\theta + \tan^5\theta)\, e^{-\tan^2\theta} d\theta$.

Solution : Let

$$I = \int_0^{\frac{\pi}{2}} (\tan^3\theta + \tan^5\theta)\, e^{-\tan^{2\theta}} d\theta$$

Let us put

$$\tan^2\theta = x$$

or

$$2 \tan\theta \sec^2\theta d\theta = dx$$

or

$$2\tan\theta\,(1 + \tan^2\theta)\, d\theta = dx$$

or

$$2\sqrt{x}\,(1+x)\, d\theta = dx$$

or

$$d\theta = \frac{dx}{2\sqrt{x}\,(1+x)}$$

when $\theta = 0$, $x = 0$ and when $\theta = \frac{\pi}{2}$, $x = \infty$

$$\therefore \qquad I = \int_0^\infty (x\sqrt{x} + x^2\sqrt{x})e^{-x}\frac{dx}{2\sqrt{x}(1+x)}$$

or
$$I = \int_0^\infty \frac{x\sqrt{x}\,(1+x)e^{-x}}{2\sqrt{x}\,(1+x)}\,dx$$

or
$$I = \frac{1}{2}\int_0^\infty xe^{-x}dx$$

or
$$I = \frac{\Gamma(2)}{2}$$

or
$$I = \frac{1}{2} \qquad [\because \Gamma(2) = 1]$$

$$\therefore \int_0^{\frac{\pi}{2}} (\tan^3\theta + \tan^5\theta)\, e^{-\tan^2\theta} d\theta = \frac{1}{2}$$

1.2. OTHER FORMS OF GAMMA FUNCTIONS

Gamma function described by the form

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx,\ n > 0$$

is known as the integral form of gamma function. Except this form there are two other forms of gamma function known as

1. Euler's form : This form of gamma function is given as

$$\Gamma(n) = \lim_{n\to\infty} \frac{1.2.3.....m}{n(n+1)....(n+m)} m^n \qquad ...(1.10)$$

Here n is neither zero nor a negative number.

2. Weierstrass' Infinite product definition : This form of gamma function is given as

$$\frac{1}{\Gamma(n)} = ne^{rn}\prod_{m=1}^{\infty}\left(1+\frac{m}{n}\right)e^{-n/m} \qquad ...(1.11)$$

where r is Euler's constant and is given as

$$r = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+....\frac{1}{m}-\ln m\right) = 0.577216$$

The integral, Euler's and Weierstrass' forms of gamma function are equivalent to one another. To prove the result consider the integral $\int_0^m \left(1-\frac{x}{m}\right)^m x^{n-1}dx$ with $m \to \infty$ such that the integral becomes

$$\int_0^m \lim_{m\to\infty}\left(1-\frac{x}{m}\right)^m x^{n-1}dx \qquad ...(1.12)$$

But $\lim\limits_{m\to\infty}\left(1-\dfrac{x}{m}\right)^m = e^{-x}$ hence equation(1.12) could be modified as

$$\int_0^\infty e^{-x}x^{n-1}dx \;=\; \Gamma(n)$$

Thus the integral $\int_0^m\left(1-\dfrac{x}{m}\right)^m x^{n-1}dx$ could be solved by substituting $\dfrac{x}{m}=t$, such that $\dfrac{dx}{m}=dt$

when $x = 0$, $t = 0$ and when $x = m$, $t=\dfrac{x}{m}=\dfrac{m}{m}=1$, so that

$$\int_0^m\left(1-\frac{x}{m}\right)^m x^{n-1}dx \;=\; \int_0^1(1-t)^m(mt)^{n-1}m\,dt = m^n\int_0^1(1-t)^m t^{n-1}dt$$

Integrating by parts one get

$$m^n\left[(1-t)^m\frac{t^n}{n}\Bigg|_0^1+\int_0^1\frac{t^n}{n}\,.\,m(1-t)^{m-1}dt\right] = m^n\left[0+\frac{m}{n}\int_0^1 t^n(1-t)^{m-1}dt\right]= m^n.\frac{m}{n}\int_0^1 t^n(1-t)^{m-1}dt$$

repeating the procedure m times, one gets

$$\int_0^m\left(1-\frac{x}{m}\right)^m x^{n-1}dx \;=\; m^n.\frac{m(m-1)(m-2).....1}{n(n+1)(n+2)....(n+m-1)}\int_0^1 t^{n+m-1}\,dt$$

$$= m^n.\frac{1.2......(m-1).m}{n(n+1)(n+2).....(n+m)}=\Gamma(n)$$

To prove Weierstrass' form consider the inverse of Euler's form, *i.e.*

$$\frac{1}{\Gamma(n)} \;=\; \frac{n(n+1)\,(n+2)....(n+m)}{1.2.3......m}\,m^{-n}$$

$$= n(1+n)\frac{(n+2)}{2}.\frac{(n+3)}{3}......\frac{(n+m)}{m}m^{-n}$$

$$= n(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)+......\left(1+\frac{n}{m}\right)e^{-n\ln m}$$

Multiply and divide the above relation with $e^{n\left(1+\frac{1}{2}+\frac{1}{3}+...\frac{1}{m}\right)}$ one gets

$$\frac{1}{\Gamma(n)} = n(1+n)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)+.....\left(1+\frac{n}{m}\right)e^{n\left(1+\frac{1}{2}+\frac{1}{3}+....\frac{1}{m}\right)}\,.\,e^{-n\left(1+\frac{1}{2}+\frac{1}{3}+....\frac{1}{m}\right)}e^{-n\ln m}$$

$$= n\left[(1+n)\, e^{-n}\left(1+\frac{n}{2}\right) e^{-n/2}\left(1+\frac{n}{3}\right) e^{-n/3}....\left(1+\frac{n}{m}\right) e^{n/m}\right] .\ e^{+n\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{m}-n\ln m\right]}$$

$$= \ ne^{rn} \prod_{m=1}^{\infty}\left(1+\frac{n}{m}\right) e^{-n/m}$$

<u>Example 1.12.</u> Prove that $\Gamma(n)\,\Gamma(-n) = -\dfrac{\pi}{n\sin n\pi}$ n is not an integer.

Solution : To start with Weierstrass infinite product of gamma function one gets

$$\frac{1}{\Gamma(n)}\cdot\frac{1}{\Gamma(-n)} = ne^{rn}\prod_{m=1}^{\infty}\left(1+\frac{n}{m}\right)e^{-n/m}\ .\ -ne^{-rn}\prod_{m=1}^{\infty}\left(1-\frac{n}{m}\right)e^{+n/m}$$

$$= -n^2\prod_{m=1}^{\infty}\left(1-\frac{n}{m}\right)\left(1+\frac{n}{m}\right)e^{-n/m}e^{n/m} \qquad ...(i)$$

$$= -n^2\prod_{m=1}^{\infty}\left(1-\frac{n^2}{m^2}\right)$$

Expanding cos kx, where k is not an integer, in fourier series in the interval $-\pi \le x \le \pi$, one gets

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx\ dx = \frac{1}{\pi}\ \frac{\sin kx}{k}\Bigg|_{-\pi}^{\pi} = \frac{1}{\pi}\ \frac{\sin k\pi+\sin k\pi}{k} = \frac{2\sin k\pi}{k\pi}$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos kx\ \cos mx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(k+m)\,x+\cos\,(k-m)\,x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(k+m)\,x}{k+m}\Bigg|_{-\pi}^{\pi} + \frac{\sin(k-m)\ x}{k-m}\Bigg|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\sin(k+m)\,\pi}{k+m}+\frac{2\sin(k-m)\,\pi}{k-m}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin k\pi\cos m\pi+\cos k\,\pi\sin m\pi}{k+m}+\frac{\sin k\,\pi\cos m\pi-\cos k\pi\sin m\pi}{k-m}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin k\pi(-1)^m+0}{k+m}+\frac{\sin k\pi(-1)^m-0}{k-m}\right]$$

$$= \frac{1}{\pi}\left[\frac{(k-m)(-1)^m \sin k\pi + (k+m)(-1)^m \sin k\pi}{k^2-m^2}\right]$$

$$\frac{1}{\pi}\left[\frac{k(-1)^m \sin k\pi - m(-1)^m \sin k\pi + k(-1)^m \sin k\pi + m(-1)^m \sin k\pi}{k^n-m^2}\right] = \frac{2k}{\pi}\frac{\sin k\pi(-1)^m}{k^2-m^2}$$

$b_n = 0$ as the function cos kx is an even function, hence

$$\cos kx = \frac{1}{k\pi}\sin k\pi + \frac{2k}{\pi}.\sum_{m=0}^{\infty}\frac{\sin k\pi(-1)^m}{k^2-m^2}\cos mx$$

Let $x = \pi$ and $k = n$, so that above equation becomes

$$\cos n\pi = \frac{1}{n\pi}\sin n\pi + \frac{2n}{\pi}\sum_{m=0}^{\infty}\frac{\sin n\pi}{n^2-m^2}(-1)^{2m}$$

or
$$\frac{\cos n\pi}{\sin n\pi} = \frac{1}{n\pi} + \frac{2n}{\pi}\sum_{m=0}^{\infty}\frac{1}{n^2-m^2}$$

Integrating the two sides with respect to n in the interval 0 to n one gets

$$\ln\frac{\sin n\pi}{n\pi} = \sum_{n=1}^{\infty}\log\left(1-\frac{n^2}{m^2}\right)$$

or
$$\frac{\sin n\pi}{n\pi} = \prod_{n=1}^{\infty}\left(1-\frac{n^2}{m^2}\right) \qquad ...(ii)$$

Comparing equation (i) and (ii) one gets

$$-n^2\frac{\sin n\pi}{n\pi} = \frac{1}{\Gamma(n)}\frac{1}{\Gamma(-n)}$$

or
$$\frac{-n \sin n\pi}{\pi} = \frac{1}{\Gamma(n)}\frac{1}{\Gamma(-n)}$$

Reverting the above relation, one gets

$$\Gamma(n)\,\Gamma(-n) = \frac{-\pi}{n \sin n\pi}$$

2. BETA FUNCTION

The Beta function is also known as Euler integral of the first kind and defined by a definite integral as

$$\boldsymbol{\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,} \quad \text{where } m > 0, n > 0 \qquad ...(2.1)$$

Note that the function depend upon m and n but not on x. It is an area function which depends upon two variables.

Thus
$$\beta(1, 1) = \int_0^1 x^0 (1-x)^0 dx = \int_0^1 dx = x]_0^1 = 1$$
$$\beta(2, 3) = \int_0^1 x^{2-1}(1-x)^{3-1} dx = \int_0^1 x(1-x)^2 \, dx, \text{ etc.}$$

2.1 PROPERTIES OF BETA (β) FUNCTION

In evaluating the integrals the following three properties are widely used

(*a*) β (*m, n*) = β (*n, m*) (Symmetry of Beta function)

(*b*) $\boldsymbol{\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx}$

(*c*) $\boldsymbol{\beta(m, n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta}$

Proof

Let us prove the properties one by one.

(*a*) To prove β (*m, n*) = β (*n, m*)

We know by definition of Beta function

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \qquad \text{[From equation (2.1)]}$$

Again we know that for definite integral

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx,$$

so using this property in the above equation, we get

$$\beta(m, n) = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1} \, dx$$
$$= \int_0^1 x^{n-1} (1-x)^{m-1} \, dx$$
$$= \beta(n, m)$$

(*b*) To prove $\boldsymbol{\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx}$

We know by definition of Beta function

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

Let $x = \frac{1}{1+y}$ or $dx = \frac{-1}{(1+y)^2} dy$

or $(1+y) = \frac{1}{x}$

Now, when $x = 0, y = \infty$

when $x = 1, y = 0$

$$\therefore \quad \beta(m, n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{m-1} \left(1 - \frac{1}{1+y}\right)^{n-1} \frac{-dy}{(1+y)^2}$$

$$= \int_{0}^{\infty} \frac{1}{(1+y)^{m-1}} \frac{y^{n-1}}{(1+y)^{n-1}} \frac{dy}{(1+y)^2}$$

[-ve sign is omitted by changing limit]

$$\therefore \quad \beta(m, n) = \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

But, $\int_a^b f(x)\, dx = \int_a^b f(y)\, dy$ [Property of definite integral]

$$\therefore \quad \beta(m, n) = \int_{0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Now using $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

(c) To prove $\beta(m, n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta$

We know that

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Let $x = \sin^2\theta$ or $dx = 2 \sin\theta \cos\theta\, d\theta$

When $x = 0, \ \theta = 0$

$x = 1, \ \theta = \frac{\pi}{2}$

$$\therefore \quad \beta(m, n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1} (1 - \sin^2\theta)^{n-1} 2\sin\theta \cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-2} (\cos\theta)^{2n-2} \sin\theta \cos\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta$$

3. RELATION BETWEEN BETA AND GAMMA FUNCTION

Beta function and Gamma function can be related by the relation

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \text{ where } m > 0, n > 0 \qquad ...(3.1)$$

In the evaluation of integrals, this relation is quite useful for finding a definite result.

Example 3.1. Prove that $\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

Solution : We know from the definition of Gamma function

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Now using the property (*c*) of the Gamma function, we write

$$\Gamma(n) = z^n \int_0^{\infty} e^{-zx} x^{n-1} dx \qquad ...(i)$$

or

$$\frac{\Gamma(n)}{z^n} = \int_0^{\infty} e^{-zx} x^{n-1} dx \qquad ...(ii)$$

Multiplying both sides by $e^{-z} z^{m-1}$ in eqn. (*i*) and then integrating with respect to z from $z = 0$ to $z = \infty$, we get

$$\Gamma(n) \int_0^{\infty} e^{-z} z^{m-1} dz = \int_0^{\infty} e^{-z} z^{m-1} \left[z^n \int_0^{\infty} e^{-zx} x^{n-1} dx \right] dz$$

or

$$\Gamma(n)\,\Gamma(m) = \int_0^{\infty}\int_0^{\infty} e^{-z(1+x)} z^{m+n-1} x^{n-1} dx\, dz$$

Since the integration limit is same, so changing the order of integration

$$\Gamma(n)\,\Gamma(m) = \int_0^{\infty} x^{n-1} \left[\int_0^{\infty} e^{-z(1+x)} z^{m+n-1} dz \right] dx$$

$$= \int_0^{\infty} x^{n-1} \left[\frac{\Gamma(m+n)}{(1+x)^{m+n}} \right] dx \qquad \text{[Using equation (ii)]}$$

or $$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta\,(m, n)$$ [Property (*b*) of Beta function]

Example 3.2. **Show that** $2\int_0^{\frac{\pi}{2}} \sin^p\theta\,\cos^q\theta\,d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$ **and hence show** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Solution : From the relation of Beta and Gamma function, we get

$$\beta\,(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad ...(i)$$

Again from the properties of Beta function (property *c*), we get

$$\beta\,(m, n) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}\,(\cos\theta)^{2n-1}\,d\theta \quad ...(ii)$$

Therefore from equation (*i*) and (*ii*)

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}\,(\cos\theta)^{2n-1}d\theta = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad ...(iii)$$

Let $2m-1 = p$ and $2n-1 = q$

or $m = \dfrac{p+1}{2}$ or $n = \dfrac{q+1}{2}$

$$\therefore \quad 2\int_0^{\frac{\pi}{2}} (\sin\theta)^p\,(\cos\theta)^q\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

Now putting $p = 0$ and $q = 0$, we get

$$2\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

or $$2\frac{\pi}{2} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \quad [\because \Gamma(1) = 1]$$

or $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Example 3.3. **Prove that** $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m),\ m > 0$

Solution : From equation (*iii*) of Example 3.2, we have

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad ...(i)$$

Putting $n = \frac{1}{2}$ in equation (*i*), we get

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}\, d\theta = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

or
$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1} d\theta = \frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} \qquad ...(ii)$$

Again putting $n = m$ in equation (*i*), we get

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2m-1} d\theta = \frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)}$$

or
$$2\int_0^{\frac{\pi}{2}} (\sin\theta\,\cos\theta)^{2m-1} d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

or
$$2\int_0^{\frac{\pi}{2}} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

or
$$\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1} d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

Let $2\theta = t$

or $2d\theta = dt$

when $\theta = 0,\ t = 0$

when $\theta = \frac{\pi}{2},\ t = \pi$

$$\therefore \quad \frac{2}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} \frac{dt}{2} = \frac{[\Gamma(m)]^2}{\Gamma(2m)}$$

$$\text{or} \quad \frac{1}{2^{2m-1}} \int_0^{\pi} (\sin \theta)^{2m-1} d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)} \qquad \left[\because \int_a^b f(t)\, dt = \int_a^b f(\theta)\, d\theta\right]$$

$$\text{or} \quad \frac{1}{2^{2m-1}} 2\int_0^{\frac{\pi}{2}} (\sin \theta)^{2m-1} d\theta = \frac{[\Gamma(m)]^2}{\Gamma(2m)} \qquad ...(iii)$$

$$\text{or} \quad \frac{1}{2^{2m-1}} \frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)} = \frac{[\Gamma(m)]^2}{\Gamma(2m)} \qquad \text{[Using equation } (ii) \text{ in equation } (iii)]$$

$$\Rightarrow \quad \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m) = \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)$$

Example 3.4. **Prove that $\beta(m, n) = \beta(m, n+1) + \beta(m+1, n)$.**

Solution : We know that

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Now

$$\beta(m, n+1) + \beta(m+1, n) = \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} + \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}$$

$$= \frac{\Gamma(m) n\, \Gamma(n)}{(m+n)\,\Gamma(m+n)} + \frac{m\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\left[\frac{n}{m+n} + \frac{m}{m+n}\right]$$

$$= \beta(m, n) \qquad \left[\because \beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

Example 3.5. **Prove that $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m, n)}{m+n}$.**

Solution : We know that

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad ...(i)$$

Now
$$\frac{\beta(m+1, n)}{m} = \frac{1}{m}\frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}$$
$$= \frac{1}{m}\frac{m\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$
$$= \frac{\beta(m, n)}{m+n} \quad ...(ii)$$

Now
$$\frac{\beta(m, n+1)}{n} = \frac{1}{n}\frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)}$$
$$= \frac{1}{n}\frac{\Gamma(m)n\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$
$$= \frac{\beta(m, n)}{m+n} \quad ...(iii)$$

∴ From equation (*i*), (*ii*) and (*iii*), we get
$$\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m,n)}{m+n}$$

Example 3.6. **Solve** $\int_0^1 x^5(1-x^3)^{10}dx$

Solution : We know that
$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad ...(i)$$

Let
$$I = \int_0^1 x^5(1-x^3)^{10}dx$$

Put $x^3 = y$ [Note that in equation (*i*) there is no power in x within the bracket]

or $3x^2\,dx = dy$

Now, when $x = 0, y = 0$

when $x = 1, y = 1$

$$\therefore \quad I = \int_0^1 (y^{\frac{1}{3}})^5 (1-y)^{10}\frac{dy}{3y^{\frac{2}{3}}}$$
$$= \frac{1}{3}\int_0^1 y^{\frac{5}{3}}y^{-\frac{2}{3}}(1-y)^{10}dy$$
$$= \frac{1}{3}\int_0^1 y(1-y)^{10}dy$$

$$= \frac{1}{3}\int_0^1 y^{2-1}(1-y)^{11-1}\,dy$$

$$= \frac{1}{3}B(2,11)$$

$$= \frac{1}{3}\frac{\Gamma(2)\,\Gamma(11)}{\Gamma(2+11)}$$

$$= \frac{1}{3}\frac{1!10!}{12!}$$

$$= \frac{1}{3}\frac{1!.10!}{12.11.10!} = \frac{1}{396}$$

SUMMARY

- Definition of Γ function $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx,\ n > 0$
- Some other form of Gamma Function

  (*i*) $\Gamma(n) = z^n\int_0^\infty e^{-zx}x^{n-1}dx$

  (*ii*) $\Gamma(n) = \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy$

  (Note that in exponential form the limit is from 0 to ∞ and in logarithm form limit is from 0 to 1. In evaluating the integrals if there is exponential term or logarithm term, then Γ function may be applied)
- Definition of β function $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m > 0, n > 0$
- Relation between β and Γ function

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

- To evaluate integrals under β function, we have to use the relation between β and Γ function to get a finite result.

QUESTIONS

MULTIPLE CHOICE QUESTIONS (MCQ's)

1. Which of the following is true?

(A) $\Gamma(n+1) = n\Gamma(n)$ for any real number

(B) $\Gamma(n) = n\Gamma(n+1)$for any real number

(C) $\Gamma(n+1) = n\Gamma(n)$ for $n > 1$

(D) $\Gamma(n) = n\Gamma(n+1)$ for $n > 1$

2. $\Gamma(n + 1) = n!$ can be used where n is ____

(A) any integer (B) a positive integer

(C) a negative integer (D) any real number

3. Which of the following is not a definition of Gamma Function?

(A) $\Gamma(n) = n!$ (B) $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

(C) $\Gamma(n + 1) = n\Gamma(n)$ (D) $\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

4. What is the value of $\Gamma\left(\frac{1}{2}\right)$?

(A) $\sqrt{\pi}$ (B) $\sqrt{\frac{\pi}{2}}$ (C) $\frac{\sqrt{\pi}}{2}$ (D) π

5. Which of the following statement is correct?

(A) $\beta(m, n) = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$ (B) $\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(C) $\beta(m, n)\Gamma(m) = \frac{\Gamma(n)}{\Gamma(m+n)}$ (D) $\beta(m, n) = \Gamma(m)\,\Gamma(n)$

6. What is the value of $\Gamma(5.5)$?

(A) $\frac{11.9.7.5}{32}\sqrt{\pi}$ (B) $\frac{9.7.5.3.1}{32}\sqrt{\pi}$ (C) $\frac{9.7.5.3.1}{64}\sqrt{\pi}$ (D) $\frac{11.9.7.5.3.1}{32}\sqrt{\pi}$

7. What is the value of $\int_0^\infty e^{-x^2}dx$?

(A) $\sqrt{\pi}$ (B) $\sqrt{\frac{\pi}{2}}$ (C) $\frac{\sqrt{\pi}}{2}$ (D) $\frac{\pi}{2}$

8. What is the value of $\int_0^1 (\log y)^8\, dy$?

(A) 5! (B) 7! (C) 8! (D) 9!

9. What is the value of $\Gamma\left(\frac{9}{4}\right)$?

(A) $54.\frac{1}{4}.\Gamma\left(\frac{1}{4}\right)$ (B) $\frac{9}{4}.\frac{5}{4}.\frac{1}{4}.\Gamma\left(\frac{1}{4}\right)$ (C) $\frac{5}{4}.\frac{1}{4}.\Gamma\left(\frac{5}{4}\right)$ (D) $\frac{7}{4}.\frac{5}{4}.\frac{1}{4}.\Gamma\left(\frac{1}{4}\right)$

10. The value of $\Gamma(0)$ is

(A) 0 (B) 1 (C) ∞ (D) $\sqrt{\pi}$

SHORT ANSWER TYPE QUESTIONS

1. Define Gamma function.

2. Define Beta function.

3. Write the relation between Gamma function and Beta function.

4. What is the value of the integral $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\; d\theta$?

5. What is the value of $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$?

LONG ANSWER TYPE QUESTIONS

1. Show that $\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$

2. Evaluate $\int_0^{\infty} \frac{1}{(1+x^4)}\,dx$

3. Evaluate $\int_0^{\infty} \frac{1}{c^x}\,dx$

4. Evaluate the Integrals :

(*a*) $\int_0^1 x^3(1-x)^5\,dx$ (*b*) $\int_0^1 x^{\frac{2}{5}}(1-x)^{-\frac{1}{2}}\,dx$

(*c*) $\int_0^1 x^{-\frac{2}{3}}(1-x)^{\frac{1}{2}}\,dx$ (*d*) $\int_0^1 x^5(1-x^3)^3\,dx$

(*e*) $\int_0^2 x^4(8-x^3)^{-\frac{1}{3}}\,dx$

5. Evaluate the integrals by using Gamma function :

(*a*) $\int_0^{\infty} \sqrt{x}e^{-x}\,dx$ (*b*) $\int_0^{\infty} x^4e^{-x^2}\,dx$

(*c*) $\int_0^1 (x\log x)^3\,dx$ (*d*) $\int_0^{\infty} \sqrt{x}e^{-3\sqrt{x}}\,dx$

(*e*) $\int_0^{\infty} x^6e^{-2x}\,dx$ (*f*) $\int_0^{\infty} e^{-x^2}\,dx$

(*g*) $\int_0^1 x^4\left[\ln\frac{1}{x}\right]^3 dx$ (*h*) $\int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)}\,dx$

HINTS/ANSWERS

MULTIPLE CHOICE QUESTIONS

1. (C)	**2.** (B)	**3.** (A)	**4.** (A)	**5.** (B)
6. (B)	**7.** (C)	**8.** (C)	**9.** (A)	**10.** (C)

Short Answer Type Questions

1. The Gamma function which is also known as Euler's integral of the second kind was first introduced by Euler in his work to generalize the factorial of non-integer values. Gamma function can be defined as (out of three different convenient definitions)

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \quad \text{Where} \quad n > 0.$$

2. The Beta function is also known as Euler integral of the first kind and defined by a definite integral as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \text{ where } m > 0, n > 0.$$

3. Beta function and Gamma function can be related by the relation

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad \text{where} \quad m > 0, \; n > 0.$$

4. Hint : Use $2m–1 = 1/2$ and $2n – 1 = –1/2$ in equation (*i*) of example 3.3.

5. Hint : Substitute $x^2 = \sin\theta$

Long Answer Type Questions

1. Hint : Put $n = 1 – m$ in $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

2. Hint : Put $x^2 = \tan\theta$

3. Hint : $\int_0^\infty \frac{1}{c^x} dx = \int_0^\infty e^{-x \log c} dx$ and put $x \log c = t$

4. Hints :

(*a*) $\beta(4,6) = \dfrac{\Gamma(4)\,\Gamma(6)}{\Gamma(4+6)}$ (*b*) $\beta\left(\frac{7}{5}, \frac{1}{2}\right)$ (*c*) $\beta\left(\frac{1}{3}, \frac{3}{2}\right)$

(*d*) Put $x^3 = t$ (*e*) Put $\dfrac{x^3}{8} = t$

5. (*a*) $\dfrac{\sqrt{3}}{2}$ (*b*) $3\dfrac{\sqrt{\pi}}{8}$

(*c*) $\dfrac{-3}{128}$ (*d*) $\dfrac{315\sqrt{\pi}}{16}$ **Hint :** $3\sqrt{x} = t$

(*e*) $\dfrac{45}{8}$ **Hint :** $2x = t$ (*f*) $\dfrac{\sqrt{\pi}}{2}$ **Hint :** $x^2 = t$

(*g*) $\dfrac{6}{625}$ **Hint :** Put $\ln\dfrac{1}{x} = \dfrac{45}{8}$ (*h*) $\dfrac{\sqrt{\pi}}{2}$ **Hint :** Put $x = e^{-z}$