2 Frobenius Method and Special Functions

This chapter has been devoted to the introduction of differential equations and their applications. These equations describe about the working of nature and help in studying the population growth, fluid motion and many more real world problems including the launching of satelittes. The solutions to differential equations are not numbers but the functions that describe the variation of the function. There are some differential equations that could not be solved using the simple methods available to solve these equations. In such a case, the solution is obtained using Power series or Frobenius method depending on the nature of the points of the differential equation.

1. DIFFERENTIAL EQUATIONS

A **differential equation** is a relationship between a function of one or more independent variable and its derivatives with respect to the independent variables. For example

$$\frac{d^2y(x)}{dx}+2x\frac{dy(x)}{dx}+2y(x) = f(x) \qquad ...(1.1)$$

$$\frac{\partial^2 y}{\partial x^2} = -\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad ...(1.2)$$

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial x\partial t}-\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = -f(x, t) \qquad ...(1.3)$$

In equation (1.1) y is a function of single variable *i.e.* x whereas in equation (1.2) and (1.3) y is a function of two variables *i.e.* x and t. These equations play a very useful role in solving physical problems and hence are of paramount importance in physics. For example

The different equation

$$\frac{d^2x}{dt^2} = -kx \qquad ...(1.4)$$

is widely used to solve the problem of simple harmonic oscillator, and the equation

$$\frac{d^2\mathrm{I}}{dt^2}+5\frac{d\mathrm{I}}{dt}+8\mathrm{I} = \mathrm{E}_0 \sin wt \qquad ...(1.5)$$

is utilized in determining the current I as a function of time t in an alternating current circuit.

Differential equations can be broadly classified into two classes :

Ordinary differential equations : A differential equation which contains a function of single independent variable and one or more of its derivatives with respect to the independent variable is called ordinary differential equation. Equation (1.1), (1.4) and (1.5) are examples of ordinary differential equations.

Partial differential equations : A differential equation which contains a function of two or more independent variables and one or more of its derivatives with respect to the independent variables is called partial differential equation. Equation (1.2) and (1.3) is an example of ordinary differential equations. Partial differential equation will be discussed in detail in chapter 3.

Before getting the solutions of differential equations, it is necessary to get introduced with the terminologies used in context to the differential equations. Thus the study of differential equations will be started after clarifying some definitions.

1.1. THE ORDER OF A DIFFERENTIAL EQUATION

The order of the highest derivative involved in the equation is called the order of the differential equation. The order of all the differential equations (1.1) – (1.5) is 2 as the highest derivative involved in all these equations is 2.

1.2. DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the exponent of the highest order derivative present in the differential equation. The degree of all the differential equations (1.1) – (1.5) is 1, and that of the differential equation

$$K^2\left(\frac{d^2y}{dx^2}\right)^2 = 1+\left(\frac{dy}{dx}\right)^3 \qquad ...(1.6)$$

is 2, the exponent of highest order of derivative, *i.e.* $\frac{d^2y}{dx^2}$.

1.3. SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is a relation between the dependent and independent variable without the involvement of its derivatives, but it should satisfy the given differential equation. There are many methods to solve the differential equations; however this chapter is devoted to the solutions of second order ordinary, homogeneous, linear differential equations of the form

$$\boldsymbol{\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\,y = 0} \qquad ...(1.7)$$

using power series method. In equation (1.7), P (x) and Q (x) are known functions of x. Equation (1.7) has two linearly independent solutions y_1 and y_2, subjected to the condition $\alpha_1 y_1 + \alpha_2 y_2 = 0$, where α_1 and α_2 should be zero for y_1 and y_2 to be linearly independent, so that the solution to equation (1.7) can be represented as

$$y = c_1y_1 + c_2y_2 \qquad ...(1.8)$$

To prove that equation (1.8) is a solution to equation (1.7), consider 1st and 2nd derivatives of equation (1.8) and substitute in equation (1.7), so that one may get

$c_1y''_1 + c_2y''_2 + P(x)(c_1y'_1 + c_2y'_2) + Q(x)(c_1 y_1 + c_2y_2) = 0$

Such that

$c_1y''_1 + P(x)c_1y'_1 + Q(x)\,c_1y_1 + c_2y''_2 + P(x)\,c_2y'_2 + Q(x)\,c_2y_2 = 0$

Or

$c_1 (y''_1 + P(x)\,y'_1 + Q(x)\,y_1) + c_2 (y''_2 + P(x)\,y'_2 + Q(x)\,y_2) = 0$ ...(1.9)

As y_1 and y_2 be the solutions of equation (1.7) hence equation (1.9) may be written as

$$c_1(0) + c_2(0) = 0$$

Proving that the general solution to equation (1.7) is given as (1.8)

2. SERIES SOLUTION OF DIFFERENTIAL EQUATION

Any arbitrary second order linear differential equation (SOLDE) of the form

$$\mathbf{P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0} \quad \text{...(2.1)}$$

can be converted into the general form of differential equation (1.7) by dividing equation (2.1) with $P_0(x)$, such that equation (2.1) can be rewritten as

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)}\frac{dy}{dx} + \frac{P_2(x)}{P_0(x)}y = 0$$

So that $P(x) = \frac{P_1(x)}{P_0(x)}$ and $Q(x) = \frac{P_2(x)}{P_0(x)}$, where $P(x)$ and $Q(x)$ may or may not be finite. Depending on the nature of $P(x)$ and $Q(x)$, the ordinary and singular points of the SOLDE can be defined.

(a) Ordinary Point of a Differential Equation

The point x_0 is said to be an **ordinary point** of the differential equation (1.7) if both $P(x)$ and $Q(x)$ are analytical (A function which is finite at every point and its neighbourhood) at $x = x_0$, *i.e.*, they are finite at $x = x_0$

(b) Singular Point of a Differential Equation

The point x_0 is said to be a singular point of the differential equation (1.7) if both $P(x)$ and $Q(x)$ or one of them fails to be analytical at $x = x_0$, *i.e.* they becomes infinite at $x = x_0$. Singular point may further be classified into two categories.

Regular Singular Point : If $x = x_0$ is a singular point of differential equation but both

$$\lim_{x \to x_0} (x - x_0)P(x)$$

and

$$\lim_{x \to x_0} (x - x_0)^2 Q(x)$$

are finite, the singular point x_0 is said to be **regular singular point** of equation (1.7)

Irregular Singular Point : If $x = x_0$ is a singular point of differential equation and both or one of the limits

$$\lim_{x \to x_0} (x - x_0)P(x)$$

and

$$\lim_{x \to x_0} (x - x_0)^2 Q(x)$$

becomes infinite, the singular point x_0 is said to be **irregular singular point** of equation (1.7). The above classification is mandatory as the solution of SOLDE depends on the nature of $P(x)$ and $Q(x)$.

Example 2.1. **Check if $x = 0$ is an analytical point or not for the differential equation.**

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0 \qquad ...(2.2)$$

Solution : Comparing equation (2.2) to equation (1.7) one gets

$$P(x) = 0 \text{ and } Q(x) = x^2 + 2$$

Which are both finite at $x = 0$ making $x = 0$ an analytical point of equation (2.2)

Example 2.2. **Check the behaviour of point $x = 0$ for the differential equation**

$$\frac{d^2y}{dx^2} - \frac{6}{x}y = 0 \qquad ...(2.3)$$

Solution : Comparing equation (2.3) to equation (1.7) one gets

$$P(x) = 0 \text{ and } Q(x) = \frac{-6}{x}$$

Here P (x) is finite for $x = 0$ but Q (x) is infinite making $x = 0$ a singular point. Thus to check the behaviour of $x = 0$ for Q (x) consider

$$\lim_{x\to 0}(x-0)^2\left(\frac{-6}{x}\right) = \lim_{x\to 0}(-6x) = 0 \to \text{finite}$$

Since the limit is finite for $x = 0$. Hence $x = 0$ is a regular singular point of equation (2.3)

Example 2.3. **For the given differential equation**

$$4x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (1-2x)y = 0 \qquad ...(2.4)$$

Check if $x = 0$ is an analytic point or not.

Solution : Converting equation (2.4) to general SOLDE one gets

$$\frac{d^2y}{dx^2} + \frac{3}{4x}\frac{dy}{dx} + \frac{(1-2x)}{4x^2}y = 0$$

Providing $P(x) = \frac{3}{4x}$ and $Q(x) = \frac{1}{4x^2} - \frac{1}{2x}$

which becomes infinite at $x = 0$, thus $x = 0$ is not an analytic point, but a singular point. To check the behaviour of singularity consider

$$\lim_{x\to 0}(x-0)\frac{3}{4x} = \lim_{x\to 0}\frac{3}{4} = \frac{3}{4} \text{ finite}$$

$$\lim_{x\to 0}(x-0)^2\left(\frac{1}{x^2} - \frac{2}{x}\right) = \lim_{x\to 0}(1-2x) = 1 = \text{finite}$$

Since the two limits are finite hence $x = 0$ is a regular singularity.

2.1. SERIES SOLUTION WHEN $x = x_0$ IS AN ORDINARY POINT (POWER SERIES METHOD)

The general series solution to equation (2.1) is given as

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad ...(2.5)$$

If $x = x_0$ is an ordinary point such that

$$y' = \frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

And
$$y'' = \frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the values of y, y', y'' in equation (2.4), one will get the result as

$$P_0(x) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + P_1(x) \sum_{n=0}^{\infty} n a_n x^{n-1} + P_2(x) \sum_{n=0}^{\infty} a_n x^n = 0$$

Simplifying the equation and comparing the coefficients of x and its exponents (powers) equal to zero, one will be able to find the values of different a_n's in terms of a_0 and a_1. Substituting the values of a_n's obtained in equation (2.5), the solution to equation (2.1) could be obtained.

Example 2.4. **Solve the SOLDE** $(1+x^2)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = 0$ **using power series method.**

Solution : The differential equation

$$(1+x^2)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = 0 \qquad ...(2.6)$$

can be rewritten as

$$\frac{d^2 y}{dx^2} + \frac{x}{(1+x^2)}\frac{dy}{dx} - \frac{1}{(1+x^2)} y = 0$$

Such that $P(x) = \frac{x}{(1+x^2)}$ and $Q(x) = \frac{x}{(1+x^2)}$ are finite for all real numbers, hence no singular point exists for the differential equation (2.6). The general series solution to the equation is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

such that
$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

And
$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the values of y, y', y'' in equation (2.6) one can obtain the relation

$$(1+x^2)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n(n-1)+n-1)a_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n^2-1)a_n x^n = 0$$

Equating the coefficients of x^n on both sides one can obtain, x^n in the first term could be achieved by replacing $n = n + 2$

$$(n+2)(n+1)\, a_{n+2} + (n^2-1)\, a_n = 0$$

Such that $$a_{n+2} = -\frac{(n^2-1)}{(n+2)(n+1)} a_n = -\frac{(n-1)}{(n+2)} a_n$$

It is known as the **recurrence relation**, useful for finding the terms of a sequence in a recursive manner. Substituting $n = 0, 1, 2, 3, 4$ ...., one can obtain the values of $a_n's$ in terms of a_0 and a_1 as

$$a_2 = \frac{1}{2}a_0, \qquad a_3 = \frac{0}{3}a_1 = 0,$$

$$a_4 = -\frac{1}{4}a_2 = -\frac{1}{4}.\frac{1}{2}a_0 \qquad a_5 = \frac{2}{5}a_3 = \frac{2}{5}.0 = 0,$$

$$a_6 = -\frac{3}{6}a_4 = (-1)^2 \frac{3}{6}.\frac{1}{4}.\frac{1}{2}a_0 \qquad a_7 = \frac{4}{7}a_5 = \frac{4}{7}.0 = 0$$

$$a_8 = -\frac{5}{8}a_6 = (-1)^3 \frac{5}{8}.\frac{3}{6}.\frac{1}{4}.\frac{1}{2}a_0 \qquad \vdots$$

Such that $$a_{2n} = (-1)^{n-1}\frac{(2n-3)!!}{2^n\, n!}a_0$$

[The double factorial (!!) also known as factorial of $(2n - 3)$ is the product of all odd integers upto $2n - 3$]

And $$a_{2n+1} = 0$$

Thus the solution to equation (2.6) is

$$y = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + ...+ a_1x + a_3x^3 + a_5x^5 + a_7x^7 + ...$$

or $$y = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} \qquad ...(2.7)$$

Substituting the value of a_{2n} and a_{2n+1} calculated above in eqn. (2.7) one obtains

or $$y = a_0 \sum_{n=0}^{\infty} (-1)^{n-1}\frac{(2n-3)!!}{2^n\, n!}x^{2n}$$

Example 2.5. Solve the SOLDE $y'' + xy' + y = 0$ using power series method.

Solution : The differential equation

$$y'' + xy' + y = 0 \qquad ...(2.8)$$

is already in its general form with P $(x) = x$ and Q $(x) = 1$ finite for all real numbers, hence no singular point exists for the differential equation (2.8). The general series solution to the equation is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

such that

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

And

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the values of y, y', y'' in equation (2.8) one can obtain the relation

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} n\left(n-1\right) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+1) a_n x^n = 0$$

Equating the coefficients of x^n on both sides one can obtain

$$(n + 2)\,(n + 1)\,a_{n+2} + (n + 1)\,a_n = 0$$

Such that

$$a_{n+2} = -\frac{n+1}{\left(n+2\right)\left(n+1\right)} a_n = -\frac{1}{\left(n+2\right)} a_n$$

Substituting n = 0, 1, 2, 3, 4,...., one can obtain the values of a_n's in terms of a_0 and a_1 as

$$a_2 = -\frac{1}{2} a_0, \qquad a_3 = -\frac{1}{3} a_1,$$

$$a_4 = -\frac{1}{4} a_2 = (-1)^2 \frac{1}{4}.\frac{1}{2} a_0 \qquad a_5 = \frac{1}{5} a_3 = \frac{1}{5.3} a_1,$$

$$a_6 = -\frac{1}{6} a_4 = (-1)^3 \frac{1}{6}.\frac{1}{4}.\frac{1}{2} a_0 \qquad a_7 = \frac{1}{7} a_5 = \frac{1}{7.5.3} a_1$$

$$\vdots \qquad \vdots$$

$$a_{2n} = \frac{(-1)^n}{2^n\, n!} a_0 \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!!} a_1$$

Thus the solution to equation (2.6) can be obtained by substituting the value a_{2n} and a_{2n+1} in eqn. (2.5) as

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!!} x^{2n+1}$$

Example 2.6. Solve the differential equation $\frac{d^2y}{dx^2} + \omega^2 y = 0$ using series method.

Solution : The differential eqn. is already in general form with P (x) = 0 and Q (x) = 1, Hence there is no singular point for the differential eqn. The general series solution of the eqn. is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

such that

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the values of y, y' and y'' in the given eqn. one gets

$$\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2} + \omega^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\Sigma(n\,(n-1)\, a_n + \omega^2 a_{n-2})\, x^{n-2} = 0$$

Equating the coefficients of x^{n-2}, one gets

$$n\,(n-1)\, a_n + \omega^2 a_{n-2} = 0$$

or

$$a_n = -\frac{\omega^2}{n(n-1)} a_{n-2}$$

Substituting $n = 2$ one get,

$$a_2 = -\frac{\omega^2}{2} a_0$$

Similarly substituting $n = 3$ in the recurrence relation of a_n one gets

$$a_3 = -\frac{\omega^2}{3.2} a_1$$

Proceeding in the same way one gets

$$a_4 = -\frac{\omega^2}{4.3} a_2 = (-1)^2 \frac{\omega^4}{4.3.2} a_0$$

$$a_5 = -\frac{\omega^2}{5.4} a_3 = (-1)^2 \frac{\omega^4}{5.4.3.2} a_1$$

$$a_6 = -\frac{\omega^2}{6.5} a_4 = (-1)^3 \frac{\omega^6}{6.5.4.3.2} a_0$$

$$a_7 = \frac{(-1)^3 \omega^6}{7.6.5.4.3.2.1} a_1$$

$$\vdots$$

$$a_{2n} = \frac{(-1)^n \omega^{2n}}{(2n)!} a_0$$

$$a_{2n+1} = \frac{(-1)^n \omega^{2n}}{(2n+1)!} a_1$$

So that the solution to the given differential eqn. could be obtained as

$$y = \sum a_n x^n = \sum_{n=0}^{\infty} (a_{2n} x^{2n} + a_{2n+1} x^{2n+1})$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n)!} a_0 x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n}}{(2n+1)!} a_1 x^{2n+1}$$

Substituting $(-1) = i^2$ and considering $a_1 = (-1)^n (i\omega a_0)$ one gets

$$y = \sum_{n=0}^{\infty} \frac{(i\omega x)^{2n}}{(2n)!} a_0 + \sum_{n=0}^{\infty} (-1)^n \frac{(i\omega x)^{2n+1}}{(2n+1)!} a_0'$$

or $$y = a_0 e^{i\omega x} = a_0 [\cos \omega x + i \sin \omega x]$$

2.2. SERIES SOLUTION WHEN $x = x_0$ IS A REGULAR SINGULAR POINT (FROBENIUS METHOD)

The general power series solution will not be the solution to equation (1.7) anymore and could be proved by rewriting the equation (1.7) as follows

$$\frac{d^2 y}{dx^2} = -P(x)\frac{dy}{dx} - Q(x)\, y \qquad ...(2.9)$$

If $y = \sum_{n=0}^{\infty} a_n x^n$ is the solution to the above equation around $x = x_0$, a regular singular point, then y, y' and y'' are analytic at $x = x_0$, which implies left hand side (LHS) of equation (2.9) is finite and analytic. On the right hand side (RHS), either P(x) or Q(x) or both fail to be analytic indicating that LHS ≠ RHS. Hence $y = \sum_{n=0}^{\infty} a_n x^n$ cannot be a solution to equation (1.7). But the solution is still achievable in terms of infinite series. The method of finding the solution in terms of infinite series for a second order linear differential equation having a regular singular point is known as method of Frobenius. The method assumes that for $x = x_0$, as a regular singular point, the solution of equation (1.7) has the general form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\alpha} \qquad ....(2.10)$$

Where α is a constant to be determined by comparing the coefficients of lowest power of $x - x_0$ on both sides subjected to the condition that $a_0 \neq 0$. This generally provides a quadratic equation in α, providing two values of α and is known as the indicial equation. The method of solution is similar to that for an ordinary point except that the values of α obtained using the **indicial equation,** sometimes also called characteristic equation, are also substituted back in the solution of equation (2.10) to obtain two different solutions.

Example 2.7. Check whether Frobenius method can be applied or not to the following equation.

$$\frac{d^2y}{dx^2} - \frac{5y}{x^3} = 0 \qquad \text{...(2.11)}$$

Solution : Comparing the given equation to equation (1.7) one gets

$$P(x) = 0, \; Q(x) = -\frac{5}{x^3}$$

Hence $x = 0$ is not an analytical point. Thus to check the nature of singularity consider

$$\lim_{x\to 0} (x-0)^2 Q(x) = \lim_{x\to 0} x^2\left(-\frac{5}{x^3}\right) = -\lim_{x\to 0} \frac{5}{x} \to \infty$$

Thus $x = 0$ is not a regular singularity, Hence Frobenius method could not be applied to solve differential equation (2.11)

Example 2.8. Solve in series the following differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1-x^2)y = 0 \qquad \text{...(2.12)}$$

Solution : The given equation can be rewritten as

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \frac{(1-x^2)}{x^2}y = 0$$

Such that $P(x) = -\frac{1}{2x}$ and $Q(x) = \frac{(1-x^2)}{x^2}$ becomes infinite at $x = 0$ making it a singular point. The next step is to check the nature of singularity using the relations

$$\lim_{x\to x_0} (x-x_0)P(x)$$

and

$$\lim_{x\to x_0} (x-x_0)^2 Q(x)$$

for the given equation. Thus

$$\lim_{x\to 0} -(x-0)\frac{1}{2x} = -1 = \textit{Finite}$$

and $$\lim_{x\to 0}(x-0)^2\frac{(1-x^2)}{x^2} = 1 = \textit{Finite}$$

Hence $x = 0$ is a regular singular point and thus the equation can be solved by using Frobenius method. The general solution to the given equation (2.12) will be considered as

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

So that $$y' = \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1}$$

and $$y'' = \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$$

Substituting the values of y, y', y'' in equation (2.12), one can obtain

$$2x^2\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)x^{n+\alpha-2} - x\sum_{n=0}^{\infty} a_n (n+\alpha)x^{n+\alpha-1} + (1-x^2)\sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$

Or $$2\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)x^{n+\alpha} - \sum_{n=0}^{\infty} a_n (n+\alpha)x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or $$\sum_{n=0}^{\infty} a_n \left(2(n+\alpha)(n+\alpha-1)-(n+\alpha)+1\right)x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or $$\sum_{n=0}^{\infty} a_n \left((n+\alpha)\left(2(n+\alpha-1)-1\right)+1\right)x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or $$\sum_{n=0}^{\infty} a_n \left((n+\alpha)(2n+2\alpha-3)+1\right)x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or $$\sum_{n=0}^{\infty} a_n \left(2(n+\alpha)-1\right)(n+\alpha-1)x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Equating the coefficients of **lowest power of x,** *i.e.*, x^{α} on both sides one can obtain

$$a_0 (2\alpha - 1)(\alpha - 1) = 0$$

As $a_0 \neq 0$, hence $(2\alpha - 1)(\alpha - 1) = 0$, which is an **indicial equation** as explained in previous section and provides the value of α as

$$\boldsymbol{\alpha = 1 \text{ or } \frac{1}{2}}$$

Comparing the coefficients of $x^{\alpha+1}$, one can obtain

$$a_1 (2(1+\alpha)-1)(1+\alpha-1) = 0$$

Or $$a_1 \alpha (2\alpha + 1) = 0$$

a_1 should be equal to zero as $\alpha = 1$ or $\frac{1}{2}$ and hence $\alpha\,(2\alpha + 1)$ can't be zero. Equating the **coefficients of $x^{n+\alpha}$,** the following recurrence relation could be observed

$$\sum_{n=0}^{\infty} a_n \left(2(n+\alpha)-1\right)(n+\alpha-1)x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or
$$a_n = \frac{a_{n-2}}{(2n+2\alpha-1)(n+\alpha-1)} \qquad \text{...(2.13)}$$

Since a_1 is zero, hence $a_3 = a_5 = a_7 = \ldots = 0$, find the values of even $a'_n s$, substitute the values of $n = 2, 4, 6, \ldots..$ in equation (2.13)

$$a_2 = \frac{1}{(2\alpha+3)(\alpha+1)} a_0$$

$$a_4 = \frac{1}{(2\alpha+7)(\alpha+3)} a_2 = \frac{1}{(2\alpha+7)(2\alpha+3)(\alpha+1)(\alpha+3)} a_0$$

$$a_6 = \frac{1}{(2\alpha+11)(\alpha+5)} a_4 = \frac{1}{(2\alpha+11)(2\alpha+7)(2\alpha+3)(\alpha+1)(\alpha+3)(\alpha+5)} a_0$$

$$a_{2n} = \frac{1}{(2\alpha+4n-1)...(2\alpha+11)(2\alpha+7)(2\alpha+3)(\alpha+1)(\alpha+3)(\alpha+5)...(\alpha+2n-1)} a_0$$

When $\alpha = 1$, the expression of a_{2n} reduces to

$$a_{2n} = \frac{1}{(4n+1)\ldots 13.9.5.2.4.6\ldots 2n} a_0 = \frac{1}{2^n\, n!(4n+1)\ldots 13.9.5} a_0$$

When $\alpha = \frac{1}{2}$, the expression of a_{2n} reduces to

$$a_{2n} = \frac{1}{4n\ldots 12.8.4.\frac{3}{2}.\frac{7}{2}.\frac{11}{2}\ldots\frac{4n-1}{2}} a_0' = \frac{1}{4^n\, n!\frac{3.7.11\ldots 4n-1}{2^n}} a_0' = \frac{1}{2^n\, n!3.7.11\ldots 4n-1} a_0'$$

[The coefficient of expansion is different from a_0 and hence is considered as a_0']

So that the generalized solution of equation (2.12) could be written as

$$y = a_0 \sum_{n=0}^{\infty} \frac{1}{2^n\, n!(4n+1)\ldots 13.9.5} x^{2n+1} + a'_0 \sum_{n=0}^{\infty} \frac{1}{2^n\, n!3.7.11\ldots 4n-1} x^{2n+\frac{1}{2}}$$

Quick Review

* Working steps for Frobenius method of solution

1. Consider $y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$ and then find y' and y''.
2. Equate the co-efficient of lowest power of x to find α
3. Equate the co-efficient of $x^{n+\alpha}$ to find the $a_n's$ for the two values of α.

Example 2.9. Discuss whether two Frobenius series solutions exist or not for the given differential equation.

$$2x^2y'' + x(x+1)y' - (\cos x)\,y = 0 \qquad \text{...(2.14)}$$

Solution : The Frobenius solutions exists for given differential equation if the difference between the roots r_1 and r_2 of the quadric equation

$$r^2 + (p(0) - 1)\,r + q(0) = 0 \qquad \text{...(2.15)}$$

is non-zero *i.e.*, $r_1 - r_2 \neq 0$, here $r^2 = x^2\dfrac{d^2y}{dx^2}$, $r = \dfrac{xdy}{dx}$ and $1 = \dfrac{dy}{dx}$.

The given eqn. (2.14) could be put in the r form by dividing it with 2 such that

$$x^2y'' + \frac{(x+1)}{2}xy' - \frac{(\cos x)}{2}y = 0$$

with $p(x) = \dfrac{x+1}{2}$ and $q(x) = -\dfrac{\cos x}{2}$ such that $p(0) = \dfrac{1}{2}$ and $q(0) = \dfrac{-1}{2}$, so that eqn. (2.15) becomes

$$r^2 + \left(\frac{1}{2} - 1\right)r - \frac{1}{2} = 0$$

or $$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$$

or $$2r^2 - r - 1 = 0$$

or $$(2r+1)(r-1) = 0$$

or $$r = 1, \frac{-1}{2}$$

so that $$r_1 - r_2 = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2} \neq 0$$

Hence two Frobenius series solutions for the given equation exist.

Example 2.10. Solve the differential equation

$$5x^2y'' + x(x+1)\,y' - y = 0 \qquad \text{...(2.16)}$$

using Frobenius method.

Solution : The given equation can be rewritten as

$$\frac{d^2y}{dx^2} + \frac{x+1}{5x}\frac{dy}{dx} - \frac{1}{5x^2}y = 0$$

such that $P(x) = \dfrac{x+1}{5x}$ and $Q(x) = -\dfrac{1}{5x^2}$ becomes infinite at $x = 0$ making it a singular point. The next step is to check the nature of singularity using the relations

$$\lim_{x \to x_0} (x - x_0)P(x) = \lim_{x \to 0} (x-0)\frac{x+1}{5x} = \lim_{x \to 0} \frac{x+1}{5} = \frac{1}{5} = \text{finite}$$

and
$$\lim_{x \to x_0} (x - x_0)^2 Q(x) = \lim_{x \to 0} (x-0)^2 \frac{1}{5x^2} = \frac{1}{5} = \text{finite}$$

Since the two limits are finite hence the differential equation (2.14) has $x = 0$ as a regular singular point and thus could be solved using Frobenius method with

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$$

Such that eqn. (2.14) could be rewritten as

$$5x^2 \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2} + x(x+1) \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1} - \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$

or
$$5 \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha} + \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha+1} + \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha} - \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$

or
$$\sum_{n=0}^{\infty} [5(n+\alpha)(n+\alpha-1) + (n+\alpha) - 1] a_n x^{n+\alpha} + \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha+1} = 0$$

or
$$\sum_{n=0}^{\infty} [(n+\alpha-1)(5(n+\alpha)+1)] a_n x^{n+\alpha} + \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha+1} = 0$$

or
$$\sum_{n=0}^{\infty} (n+\alpha-1)(5n+5\alpha+1) a_n x^{n+\alpha} + \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha+1} = 0$$

Comparing the coefficients of $x^{n+\alpha}$ on both sides one gets

$$(n+\alpha-1)(5n+5\alpha+1) a_n + (n+\alpha-1) a_{n-1} = 0$$

or
$$a_n = -\frac{(n+\alpha-1)}{(n+\alpha-1)(5n+5\alpha+1)} a_{n-1} = -\frac{1}{5n+5\alpha+1} a_{n-1}$$

Comparing the coefficients of x^{α}, one gets

$$(\alpha-1)(5\alpha+1) a_0 = 0$$

as $a_0 \neq 0$ Hence $(\alpha - 1)(5\alpha + 1) = 0$

$$\Rightarrow \quad \alpha = 1, -1/5$$

Comparing the coefficients of $x^{\alpha+1}$ one gets

$$\alpha(5\alpha+6) a_1 + \alpha a_0 = 0$$

or
$$a_1 = -\frac{a_0}{5\alpha+6}$$

$$a_2 = -\frac{1}{5\alpha+11}a_1 = \frac{(-1)^2 a_0}{(5\alpha+6)(5\alpha+11)}$$

$$a_3 = \frac{-1}{5\alpha+15}a_2 = \frac{(-1)^3 a_0}{(5\alpha+6)(5\alpha+11)(5\alpha+16)}$$

$$\vdots$$

$$a_n = \frac{(-1)^n a_0}{(5\alpha+6)(5\alpha+11)(5\alpha+16)\ldots.(5\alpha+5n+1)}$$

for $\alpha = 1$, a_n becomes

$$a_n = \frac{(-1)^n a_0}{11.16.21\ldots.(5n+6)}$$

and for $\alpha = -\frac{1}{5}$ a_n becomes

$$a_n = \frac{(-1)^n a_0'}{5.10.15\ldots..5n} = \frac{(-1)^n a_0'}{5^n\,(1.2.3\ldots.n)} = \frac{(-1)^n a_0'}{5^n n!}$$

So that the solutions to eqn. (2.14) is given as

$$y = \sum_{n=0}^{\infty}\frac{(-1)^n a_0}{11.16.21\ldots..5n+6}\,x^{n+1} + \sum_{n=0}^{\infty}\frac{(-1)^n a_0'}{5^n n!}\,x^{n-\frac{1}{5}}$$

or
$$y = \sum_{n=0}^{\infty}\frac{(-1)^n a_0}{11.16.21\ldots..5n+6}\,x^{n+1} + x^{-1/5}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-x}{5}\right)^n$$

or
$$y = \sum_{n=0}^{\infty}\frac{(-1)^n a_0}{11.16.21\ldots..5n+6}\,x^{n+1} + x^{-1/5}e^{-x/5} \qquad \left[e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right]$$

3. LEGENDRE'S DIFFERENTIAL EQUATION AND IT'S SOLUTION

Legendre's differential equation is a particular 2nd order linear differential equation which has a wide application in different branches of physics.

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \qquad \ldots(3.1)$$

is called Legendre's equation of l^{th} order and its solutions are called Legendre's Polynomials. The equation is of considerable importance in solving the spherical harmonics in quantum mechanics, nuclear physics, etc. The equation can be rewritten as

$$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{l(l+1)}{(1-x^2)}y = 0$$

Such that $P(x) = -\frac{2x}{(1-x^2)}$ and $Q(x) = \frac{l(l+1)}{(1-x^2)}$ becomes infinite at $x = \pm 1$ making it a singular point. The next step is to check the nature of singularity using the relations

$$\lim_{x \to x_0} (x - x_0) P(x)$$

and

$$\lim_{x \to x_0} (x - x_0)^2 Q(x)$$

for Legendre's equation. Thus

$$\lim_{x \to \pm 1} (1 \pm x) \frac{2x}{(1-x^2)} = 1 = \textit{Finite}$$

and

$$\lim_{x \to \pm 1} (x \pm 1)^2 \frac{l(l+1)}{1-x^2} = 0 = \textit{Finite}$$

Hence $x = \pm 1$ is a regular singular point and thus the equation can be solved using Frobenius method in series of ascending or descending powers of x. However the solution in descending powers of x is more useful and hence the Legendre's equation will be solved in descending powers of x in this chapter. The general solution of equation (3.1) will be considered as

$$y = \sum_{n=0}^{\infty} a_n x^{\alpha - n}$$

So that

$$y' = \sum_{n=0}^{\infty} a_n (\alpha - n) x^{\alpha - n - 1}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n (\alpha - n)(\alpha - n - 1) x^{\alpha - n - 2}$$

Substituting the values of y, y', y'' in equation (3.1), one can obtain

$$(1-x^2) \sum_{n=0}^{\infty} a_n (\alpha - n)(\alpha - n - 1) x^{\alpha-n-2} - 2x \sum_{n=0}^{\infty} a_n (\alpha - n) x^{\alpha-n-1} + l(l+1) \sum_{n=0}^{\infty} a_n x^{\alpha-n} = 0$$

Or
$$\sum_{n=0}^{\infty} a_n (\alpha - n)(\alpha - n - 1) x^{\alpha-n-2} - \sum_{n=0}^{\infty} a_n (\alpha - n)(\alpha - n - 1) x^{\alpha-n} - 2 \sum_{n=0}^{\infty} a_n (\alpha - n) x^{\alpha-n}$$
$$+ l(l+1) \sum_{n=0}^{\infty} a_n x^{\alpha-n} = 0$$

Or
$$\sum_{n=0}^{\infty} a_n (\alpha - n)(\alpha - n - 1) x^{\alpha-n-2} - \sum_{n=0}^{\infty} a_n \left\{(\alpha - n)(\alpha - n - 1) + 2(\alpha - n) - l(l+1)\right\} x^{\alpha-n} = 0$$

Or
$$\sum_{n=0}^{\infty} a_n (\alpha - n)(\alpha - n - 1) x^{\alpha-n-2} - \sum_{n=0}^{\infty} a_n \left\{(\alpha - n)(\alpha - n + 1) - l(l+1)\right\} x^{\alpha-n} = 0$$

Or $\sum_{n=0}^{\infty} a_n (\alpha-n)(\alpha-n-1)x^{\alpha-n-2} - \sum_{n=0}^{\infty} a_n \{(\alpha-n-l)(\alpha-n+l+1)\} x^{\alpha-n} = 0$

Equating the coefficients of highest power of x, *i.e.* x^{α} (by substituting $n = 0$ in second term of the above eqn.) on both sides one can obtain

$$a_0 \{(\alpha - l)(\alpha + l + 1)\} = 0$$

As $a_0 \neq 0$, hence $(\alpha - l)(\alpha + l + 1) = 0$, which is an **indicial equation** and provides the value of α as

$$\boldsymbol{\alpha = l \text{ or } -l-1}$$

Comparing the coefficients of $x^{\alpha-1}$, one can obtain

$$a_1 \{(\alpha - 1 - l)(\alpha + l)\} = 0$$

a_1 should be equal to zero as the above equation provides $\alpha = l + 1$ or $-l$, which is not possible as the values α have already been found. Equating the coefficients of $x^{\alpha-n}$, the following recurrence relation could be observed

$a_{n-2} (\alpha - n + 2)(\alpha - n + 1) - a_n (\alpha - n \ \ l)(\alpha - n + l + 1)) = 0$

Or $$a_n = \frac{(\alpha-n+2)(\alpha-n+1)}{(\alpha-n-l)(\alpha-n+l+1)} a_{n-2} = -\frac{(\alpha-n+2)(\alpha-n+1)}{(l+n-\alpha)(l+\alpha-n+1)} a_{n-2} \quad \text{...(3.2)}$$

Since a_1 is zero, hence $a_3 = a_5 = a_7 = ... = 0$, to find the values of even $a_n's$, substitute the values of $n = 2, 4, 6$ ..... in equation (3.2)

$$a_2 = \frac{-\alpha(\alpha-1)}{(l+2-\alpha)(l+\alpha-1)} a_0$$

$$a_4 = \frac{-(\alpha-2)(\alpha-3)}{(l+4-\alpha)(l+\alpha-3)} a_2 = (-1)^2 \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(l+4-\alpha)(l+2-\alpha)(l+\alpha-1)(l+\alpha-3)} a_0$$

$$a_6 = \frac{-(\alpha-4)(\alpha-5)}{(l+6-\alpha)(l+\alpha-5)} a_4 = (-1)^3 \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)}{(l+6-\alpha)(l+4-\alpha)(l+2-\alpha)(l+\alpha-1)(l+\alpha-3)(l+\alpha-5)} a_0$$

$$a_{2n} = (-1)^n \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)....(\alpha-2n+1)}{(l+2n-\alpha)...(l+4-\alpha)(l+2-\alpha)(l+\alpha-1)(l+\alpha-3)...(l+\alpha-2n+1)} a_0$$

When $\alpha = l$, the expression of a_{2n} redues to

$$a_{2n} = (-1)^n \frac{l(l-1)(l-2)(l-3)....(l-2n+1)}{(2n)...4.2.\,(2l-1)(2l-3)....(2l-2n+1)} a_0$$

$$= (-1)^n \frac{l(l-1)(l-2)(l-3)....(l-2n+1)}{2^n\, n!(2l-1)(2l-3)....(2l-2n+1)} a_0$$

When $\alpha = -l - 1$, the expression of a_{2n} reduces to

$$a_{2n} = (-1)^n \frac{(l+1)(l+2)(l+3)(l+4)...(l-2n)}{(2l+2n+1)...\,(2l+5)(2l+3)2.4...2n} a'_0$$

$$= \frac{(l+1)(l+2)(l+3)(l+4)...(l-2n)}{2^n n!(2l+3)(2l+5)...(2l+2n+1)} a_0'$$

[a_0' is diffeent than a_0]

So that the generalized solution of Legendre's equation could be written as

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{l(l-1)(l-2)(l-3)...(l-2n+1)}{2^n n!(2l-1)(2l-3)....(2l-2n+1)} x^{l-2n}$$

$$+ a_0' \sum_{n=0}^{\infty} \frac{(l+1)(l+2)(l+3)(l+4)...(l-2n)}{2^n n!(2l+3)(2l+5)....(2l+2n+1)} x^{-l-2n-1} \quad ...(3.3)$$

or $y = a_0 P_l(x) + a_0' Q_l(x)$

Here $P_l(x)$ is known as Legendre's function of first kind and $Q_l(x)$ is Legendre's function of second kind.

Quick Review

* Working steps for solution of Legendre's differential equation

1. Consider $y = \sum_{n=0}^{\infty} a_n x^{\alpha-n}$ and find y' and y''.
2. Equate the highest power of x to find the values of α
3. Equate the co-efficient of $x^{\alpha-n}$ to find the a_n's for the two values of α.

3.1. LEGENDRE'S POLYNOMIALS $P_L(X)$

Legendre's polynomials $P_l(x)$ are the polynomials satisfying the Legendre's differential equation. The first part of equation (3.3) is known as Legendre's polynomials of first kind and more generally Legendre polynomials and a few of these polynomials are

$$\left.\begin{aligned} &P_l(1) = 1, P_0(x) = 1 \\ &P_1(x) = x \\ &P_2(x) = \frac{1}{2}(3x^2 - 1) \\ &P_3(x) = \frac{1}{2}(5x^3 - 3x) \\ &P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}\right\} \quad ...(3.4)$$

These polynomials have been derived using generating function of Legendre's function in next sections.

3.2. RODRIGUE'S FORMULA

The result

$$\mathbf{P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l} \quad ...(3.5)$$

is known as Rodrigue's formula to represent Legendre's polynomials.

To prove Rodrigue's formula, consider

$$F = (x^2 - 1)^l \quad \text{...(3.6)}$$

Differentiating expression (3.6) with respect to x, one can obtain

$$\frac{dF}{dx} = 2xl\,(x^2 - 1)^{l-1}$$

Or
$$\left(x^2-1\right)\frac{dF}{dx} = 2xl\,(x^2-1)^{l-1}\,(x^2-1) = 2xl\,(x^2-1)^l$$

Or
$$\left(x^2-1\right)\frac{dF}{dx} = 2xl\,F \quad \text{...(3.7)}$$

Differentiating equation (3.7) $(l + 1)$ times using Libnitz formula, obtain

$$\left(x^2-1\right)\frac{d^{l+2}F}{dx^{l+2}} + 2x\left(l+1\right)\frac{d^{l+1}F}{dx^{l+1}} + l\left(l+1\right)\frac{d^lF}{dx^l} = 2xl\frac{d^{l+1}F}{dx^{l+1}} + 2l\left(l+1\right)\frac{d^lF}{dx^l}$$

Or
$$\left(x^2-1\right)\frac{d^{l+2}F}{dx^{l+2}} + 2x\left(l+1\right)\frac{d^{l+1}F}{dx^{l+1}} + l\left(l+1\right)\frac{d^lF}{dx^l} - 2xl\frac{d^{l+1}F}{dx^{l+1}} - 2l\left(l+1\right)\frac{d^lF}{dx^l} = 0$$

Or
$$\left(x^2-1\right)\frac{d^{l+2}F}{dx^{l+2}} + 2x\frac{d^{l+1}F}{dx^{l+1}} - l\left(l+1\right)\frac{d^lF}{dx^l} = 0$$

Or
$$\left(1-x^2\right)\frac{d^l}{dx^l}\left(\frac{d^lF}{dx^l}\right) - 2x\frac{d}{dx}\left(\frac{d^lF}{dx^l}\right) + l\left(l+1\right)\frac{d^lF}{dx^l} = 0$$

Thus
$$\left(\frac{d^lF}{dx^l}\right) = \left(\frac{d^l\,(x^2-1)^l}{dx^l}\right) \quad \text{[From equation (3.6)]}$$

is a solution to Legendre's differential equation. $P_l(x)$ is also a solution to Legendre's differential equation, hence

$$P_l(x) = C\left(\frac{d^l\,(x^2-1)^l}{dx^l}\right)$$

The constant of proportionality can be determined by evaluating $\left(\frac{d^l\,(x^2-1)^l}{dx^l}\right)$ at $x = 1$

$$\frac{d^l\,(x^2-1)^l}{dx^l} = \frac{d^l}{dx^l}\left(x-1\right)^l\left(x+1\right)^l$$

Using Libnitz formula to differentiate LHS one can have

$$\frac{d^l\,(x^2-1)^l}{dx^l} = (x+1)^l\frac{d^l}{dx^l}(x-1)^l + l\frac{d}{dx}(x+1)^l\frac{d^{l-1}}{dx^{l-1}}(x-1)^l$$

$$+ l(l-1)\frac{d^2}{dx^2}(x+1)^l\frac{d^{l-2}}{dx^{l-2}}(x-1)^l + ... + l\frac{d^{l-1}}{dx^{l-1}}(x+1)^l\frac{d}{dx}(x-1)^l + (x-1)^l\frac{d^l}{dx^l}(x+1)^l$$

$$\frac{d^l (x^2-1)^l}{dx^l} = l!\,(x+1)^l + l^2 (x+1)^{l-1}\frac{l!}{2}(x-1) + l^2 (l-1)^2 (x+1)^{l-2}\frac{l!}{3!}(x-1)^2 +$$

$$...+l^2 (x-1)^{1-1}\frac{l!}{2}(x+1) + (x-1)^l l!$$

Substituting $x = 1$, one can have

$$\frac{d^l (x^2-1)^l}{dx^l} = l!\,(1+1)^l + l^2 (1+1)^{l-1}\frac{l!}{2}(1-1) + l^2 (l-1)^2 (1+1)^{l-2}$$

$$\frac{l!}{3!}(1-1)^2 +...+l^2 (1-1)^{1-1}\frac{l!}{2}(1+1) + (1-1)^l l!$$

Or $$\left.\frac{d^l (x^2-1)^l}{dx^l}\right|_{x=1} = 2^l\, l!$$

Hence $$\mathbf{P}_l(x) = \frac{1}{2^l\, l!}\left(\frac{d^l (x^2-1)^l}{dx^l}\right)$$

Additional Information (Libnitz Formula of nth order differentiation)

The Leibniz formula reveals the nth order differentiation of the product of two functions. Assume that $u(x)$ and $v(x)$ are the functions of x, having derivatives up to nth order. The first order differentiation of the function is given as

$$\frac{du(x)v(x)}{dx} = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}$$

The second differentiation of the function yields

$$\frac{d^2u(x)v(x)}{dx^2} = u(x)\frac{d^2v(x)}{dx^2} + v(x)\frac{d^2u(x)}{dx^2} + 2\frac{dv(x)}{dx}\frac{du(x)}{dx}$$

Likewise the third derivative is given as

$$\frac{d^3u(x)v(x)}{dx^3} = u(x)\frac{d^3v(x)}{dx^3} + v(x)\frac{d^3u(x)}{dx^3} + 3\frac{d^2v(x)}{dx^2}\frac{du(x)}{dx} + 3\frac{d^2u(x)}{dx^2}\frac{dv}{dx}$$

Proceeding in a similar way, one may obtain

$$\frac{d^nu(x)v(x)}{dx^n} = u(x)\frac{d^nv(x)}{dx^n} + n\frac{du(x)}{dx}\frac{d^{n-1}v(x)}{dx^{n-1}} + n(n-1)\frac{d^2u(x)}{dx^2}.\frac{d^{n-2}v(x)}{dx^{n-2}} + ..$$

$$...+n\frac{d^{n-1}u(x)}{dx^{n-1}}\frac{dv(x)}{dx} + v(x)\frac{d^nu(x)}{dx^n}$$

This formula is called Libnitz Formula

Example 3.1. **Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials**

Solution : The highest power in the expression is 3, used in $P_3(x)$, thus

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x \qquad \left[P_3(x) = \frac{1}{2}(5x^3 - 3x)\right]$$

So
$$f(x) = x^3 - 5x^2 + x + 2 = \frac{2}{5}P_3(x) + \frac{3}{5}x - 5x^2 + x + 2 = \frac{2}{5}P_3(x) - 5x^2 + \frac{8}{5}x + 2$$

$$= \frac{2}{5}P_3(x) - 5\left(\frac{2}{3}P_2(x) + \frac{1}{3}\right) + \frac{8}{5}x + 2 = \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}x + \frac{1}{3}$$

$$\left[\text{Here } P_2(x) = \frac{1}{2}(3x^2 - 1)\right]$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$

3.3. GENERATING FUNCTION OF LEGENDRE'S POLYNOMIALS

Generating functions are useful mathematical tools to represent an infinite sequence. It is a single function which encodes the sequence. The generating function of Legendre's polynomial is

$$\boldsymbol{g(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l, (t < l)} \qquad ...(3.8)$$

Thus $P_l(x)$ is a coefficient of t^l in the expansion of $g(x, t)$. To prove it consider the binomial expansion of $g(x, t)$

$$(1 - 2xt + t^2)^{-1/2} = [1 - (2xt - t^2)]^{-1/2} = 1 + \frac{1}{2}(2xt - t^2) + \frac{3}{8}(2xt - t^2)^2 + \frac{5}{16}(2xt - t^2)^3 + \ldots$$

Or $$(1 - 2xt + t^2)^{-1/2} = 1 + xt - \frac{t^2}{2} + \frac{3}{2}x^2t^2 + \frac{3}{8}t^4 - \frac{3}{2}xt^3 + \frac{5}{2}x^3t^3 - t^6 - \frac{15}{4}x^2t^4 + \frac{15}{8}xt^5 \ldots$$

Or $$(1 - 2xt + t^2)^{-1/2} = 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \frac{1}{2}(5x^2 - 3x)t^3 + \frac{1}{8}(35x^4 - 30x^2 + 3)t^4 + \ldots$$

Or $$(1 - 2xt + t^2)^{-1/2} = 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \frac{1}{2}(5x^2 - 3x)t^3 + \frac{1}{8}(35x^4 - 30x^2 + 3)t^4 + \ldots$$

Or $$(1 - 2xt + t^2)^{-1/2} = P_0(x) + P_1(x)t + P_2(x)t^2 + P_3(x)t^3 + P_4(x)t^4 + \ldots = \sum_{l=0}^{\infty} P_l(x)t^l$$

Example 3.2. **Show that $P_l(1) = 1$**

Solution : Consider the generating function of $P_l(x)$ and let $x = 1$, such that

$$\frac{1}{\sqrt{1 - 2t + t^2}} = \sum_{l=0}^{\infty} P_l(1)t^l$$

Or
$$(1 - t)^{-1} = \sum_{l=0}^{\infty} P_l(1)t^l$$

Or $$1 + t + t^2 + t^3 + ... = \sum_{l=0}^{\infty} P_l(1) t^l$$

Or $$\sum_{l=0}^{\infty} t^l = \sum_{l=0}^{\infty} P_l(1) t^l$$

Comparing both sides, one gets

$$P_l(1) = 1$$

Example 3.3. Show that $P_l(-x) = (-1)^l P_l(x)$

Solution : Consider the generating function of $P_l(x)$ and changing t to $-t$, equation (3.8) can be rewritten as

$$\frac{1}{\sqrt{1+2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)(-t)^l$$

Or it can be rearranged as

$$\frac{1}{\sqrt{1-2(-x)t+t^2}} = \sum_{l=0}^{\infty} (-1)^l P_l(x) t^l$$

Or $$\sum_{l=0}^{\infty} P_l(-x) t^l = \sum_{l=0}^{\infty} (-1)^l P_l(x) t^l$$

Comparing the two sides one may get

$$P_l(-x) = (-1)^l P_l(x)$$

Example 3.4. Show that $P_0(x) = 1$

Solution : Using the generating function relation (3.8) and substituting $l = 0$, one gets

$$\frac{1}{\sqrt{1+2xt+t^2}} = P_0(x) t^0$$

Expanding the left hand side and comparing the coefficients of t^0 one gets

$$P_0(x) t^0 = 1 + xt - \frac{t^2}{2} + \frac{3}{2}x^2t^2 + \frac{3}{8}t^4 - \frac{3}{2}xt^3 + \frac{5}{2}x^3t^3 - t^6 - \frac{15}{4}x^2t^4 + \frac{15}{8}xt^5 \ldots$$

Comparing the coefficient of t^0 on both sides, one gets

$\therefore$ $P_0(x) = 1$

3.4. RECURRENCE RELATIONS

Various recurrence relations of the Legendre polynomials can be obtained from the generating function $(1 + 2xt + t^2)^{-1/2}$. The recurrence relations of Legendre's polynomials along with their establishment from the generating function have been discussed one by one

(*a*) $$(l+1) P_{l+1}(x) = (2l+1) x P_l(x) - l P_{l-1}(x)$$

To prove it consider the differentiation of both sides of equation (3.8) with respect to t, such that

$$\frac{d}{dt}(1-2xt+t^2)^{-1/2} = \frac{d}{dt}\sum_{l=0}^{\infty} P_l(x) t^l$$

Or $$-\frac{1}{2}(-2x+2t)(1-2xt+t^2)^{-3/2} = \sum_{l=0}^{\infty} l\,P_l(x)t^{l-1}$$

Or $$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{l=0}^{\infty} lP_l(x)t^{l-1} \qquad ...(3.9)$$

Multiplying the two sides with $(1 - 2xt + t^2)$, one can get

$$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{l=0}^{\infty} l\,P_l(x)t^{l-1}$$

Or $$(x-t)\sum_{l=0}^{\infty} P_l(x)t^l = \sum_{l=0}^{\infty} l\,P_l(x)t^{l-1} - 2x\sum_{l=0}^{\infty} l\,P_l(x)t^l + \sum_{l=0}^{\infty} l\,P_l(x)t^{l+1}$$

Or $$x\sum_{l=0}^{\infty} P_l(x)t^l - \sum_{l=0}^{\infty} P_l(x)t^{l+1} = \sum_{l=0}^{\infty} l\,P_l(x)t^{l-1} - 2x\sum_{l=0}^{\infty} l\,P_l(x)t^l + \sum_{l=0}^{\infty} l\,P_l(x)t^{l+1}$$

Equating the coefficients of t^l on both sides, one can get

$$xP_l(x) - P_{l-1}(x) = (l+1)P_{l+1}(x) - 2xl\,P_l(x) + (l-1)P_{l-1}(x)$$

Rearranging the terms one will be able to achieve the result

$$\mathbf{(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)} \qquad ...(3.10)$$

(*b*) $$\mathbf{P'_{l+1}(x) = P_l(x) + 2xP'_l(x) - P'_{l-1}(x)}$$

To prove it consider the differentiation of both sides of equation (3.8) with respect to x, such that

$$\frac{d}{dx}(1-2xt+t^2)^{-1/2} = \frac{d}{dx}\sum_{l=0}^{\infty} P_l(x)t^l$$

$$t(1-2xt+t^2)^{-3/2} = \sum_{l=0}^{\infty} P_l'(x)t^l \qquad ...(3.11)$$

Multiplying the two sides with $(1 - 2xt + t^2)$, one can get

$$t(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{l=0}^{\infty} P_l'(x)t^l$$

$$\sum_{l=0}^{\infty} P_l(x)t^{l+1} = \sum_{l=0}^{\infty} P_l'(x)t^l - 2x\sum_{l=0}^{\infty} P_l'(x)t^{l+1} + \sum_{l=0}^{\infty} P_l'(x)t^{l+2}$$

Comparing the coefficients of t^l on both sides one may get

$$P_{l-1}(x) = P'_l(x) - 2xP'_{l-1}(x) + P'_{l-2}(x)$$

Rearranging the terms one will be able to achieve the result

$$P'_l(x) = P_{l-1}(x) + 2xP'_{l-1}(x) - P'_{l-2}(x)$$

Replacing l by $l + 1$, the required recurrence relation can be established as

$$\mathbf{P'_{l+1}(x) = P_l(x) + 2xP'_l(x) - P_{l-1}(x)} \qquad \text{...(3.12)}$$

(c) $$\mathbf{l\,P_l(x) = xP'_l(x) - P'_{l-1}(x)}$$

Dividing equation (3.9) with (3.11), one can achieve

$$\frac{(x-t)}{t} = \frac{\sum_{l=0}^{\infty} l\,P_l(x)t^{l-1}}{\sum_{l=0}^{\infty} P_l'(x)t^l}$$

Or $$(x-t)\sum_{l=0}^{\infty} P_l'(x)t^l = t\sum_{l=0}^{\infty} lP_l(x)t^{l-1}$$

Or $$x\sum_{l=0}^{\infty} P_l'(x)t^l - \sum_{l=0}^{\infty} P_l'(x)t^{l+1} = \sum_{l=0}^{\infty} l\,P_l(x)t^l$$

Comparing the coefficients of t^l on both sides one may get

$$\mathbf{xP'_l(x) - P'_{(l-1)}(x) = l\,P_l(x)} \qquad \text{...(3.13)}$$

(d) $$\mathbf{xP'_{l-1}(x) - l\,P_{l-1}(x) = P'_l(x)}$$

Differentiating recurrence relation (3.10) with respect to x results in

$$(l+1)P'_{(l+1)}(x) = (2l+1)\,xP'_l(x) + (2l+1)\,P_l(x) - l\,P'_{l-1}(x)$$

Substituting the value of $P'(l-1)(x)$ from recurrence relation b in the above equation, one can achieve

$$(l+1)\,P'_{l+1}(x) = (2l+1)\,xP'_l(x) + (2l+1)\,P_l(x) - l\,(P_l(x) + 2xP'_l(x) - P'_{l+1}(x))$$

Or $$(l+1)\,P'_{l+1}(x) = (2l+1)\,xP'_l(x) + (2l+1)\,P_l(x) - l\,P_l(x) - 2xl\,P'_l(x) + l\,P'_{l+1}(x)$$

Or $$P'_{l+1}(x) = xP'_l(x) + (l+1)\,P_l(x)$$

Replacing l by $l - 1$ and rearranging the terms the required recurrence relation can be established as

$$\mathbf{xP'_{l-1}(x) - l\,P_{l-1}(x) = P'l(x)} \qquad \text{...(3.14)}$$

(e) $$\mathbf{(x^2-1)\,P'_l(x) = xlP_l(x) - l\,P_{l-1}(x)}$$

Multiplying recurrence relation (3.13) with x one can obtain

$$x^2\,P'_l(x) - x\,P'_{l-1}(x) = xl\,P'_l(x) \qquad \text{...(3.15)}$$

Substituting the value of $x\,P'_{l-1}(x)$ from recurrence relation (3.14) in equation (3.15), the equation can be rewritten as

$$x^2P'_l(x) - l\,P_{l-1}(x) - P'_l(x) = xl\,P'_l(x)$$

Rearranging the terms one can obtain

$$(x^2-1)\,P'_l(x) = xl\,P_l(x) - l\,P_{l-1}(x)$$

Example 3.5. Show that $P_1(x) = x$

Solution : Using the recurrence relation (3.10) and substituting $l = 0$, one gets

$$P_1(x) = xP_0(x) - 0.P_{-1}(x)$$

Substituting $P_0(x) = 1$, one gets

$$P_1(x) = x$$

Example 3.6. Derive the expression of $P_2(x)$ using recurrence relation (3.10)

Solution: Using the recurrence relation (3.10) and substituting $l = 1$, one gets

$$2P_2(x) = 3x\,P_1(x) - P_0(x)$$

Substituting $P_0(x) = 1$, and $P_1(x) = x$ one gets

$$2P_2(x) = 3x^2 - 1$$

Or
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

3.5. ORTHOGONALITY OF LEGENDRE'S POLYNOMIALS

The Legendre's polynomials satisfy the following orthogonality relation

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = \frac{2}{2l+1}\delta_{lm} \qquad ...(3.16)$$

Here δ_{lm} is a kronecker delta function and is one if the two indices, *i.e.* l and m are same and is zero if the two indices are not same, *i.e.* $l \neq m$

This relation can be proved as follows

$P_l(x)$ and $P_m(x)$ are the solutions of the Legendre's equations given as

$$(1-x^2)\frac{d^2P_l(x)}{dx^2} - 2x\frac{dP_l(x)}{dx} + l(l+1)P_l(x) = 0 \qquad ...(3.17)$$

$$(1-x^2)\frac{d^2P_m(x)}{dx^2} - 2x\frac{d\,P_m(x)}{dx} + m(m+1)P_m(x) = 0 \qquad ...(3.18)$$

Multiplying equation (3.17) with $P_m(x)$ and equation (3.18) with $P_l(x)$ and subtracting, one can achieve the following result

$$(1-x^2)\left(P_m(x)\frac{d^2P_l(x)}{dx^2} - P_l(x)\frac{d^2P_m(x)}{dx^2}\right) - 2x\left(P_m(x)\frac{d\,P_l(x)}{dx} - P_l(x)\frac{d\,P_m(x)}{dx}\right)$$
$$+ (l(l+1) - m(m+1))\,P_l(x)\,P_m(x) = 0$$

$$\frac{d}{dx}\left(P_m(x)(1-x^2)\frac{d\,P_l(x)}{dx} - P_m(x)(1-x^2)\frac{d\,P_l(x)}{dx}\right) = (m(m+1) - l(l+1))\,P_l(x)\,P_m(x)$$

Integrating the above equation with respect to x between the limits –1 to 1, one can obtain

$$\left. P_m(x)(1-x^2)\frac{d\,P_l(x)}{dx} - P_m(x)(1-x^2)\frac{d\,P_l(x)}{dx}\right|_{-1}^{1} = (m(m+1) - l(l+1)\int_{-1}^{1} P_l(x)P_m(x)\,dx$$

It results in

$$(m(m+1) - l(l+1))\int_{-1}^{1} P_l(x)P_m(x)\,dx = 0$$

$$(m-l)(m+l+1)\int_{-1}^{1} P_l(x)P_m(x)dx = 0$$

Thus there are two conclusions from the above relation, one is either $(m - l)(m + l + 1)$ is zero or $\int_{-1}^{1} P_l(x)P_m(x)dx$ is zero, that leads to the conclusion that, $\int_{-1}^{1} P_l(x)P_m(x)dx = 0$, when $l \neq m$, the second option $m = -(l + 1)$ is not possible as m cannot be negative. When $l = m$, the relation can be proved using Rodrigue's formula (3.5) by considering the integral $\int_{-1}^{1} P_l^2(x)\, dx$ as

$$\int_{-1}^{1} P_l^2(x)dx = \frac{1}{(2^l l!)^2}\int_{-1}^{1}\left(\frac{d^l(x^2-1)^l}{dx^l}\right)\left(\frac{d^l(x^2-1)^1}{dx^l}\right)dx$$

Integrating by parts the RHS, one can obtain

$$\frac{1}{(2^l l!)^2}\int_{-1}^{1}\left(\frac{d^l(x^2-1)^l}{dx^l}\right)\left(\frac{d^l(x^2-1)^l}{dx^l}\right)dx = \frac{1}{(2^l l!)^2}\left[\left(\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}\right)\left(\frac{d^l(x^2-1)^l}{dx^l}\right)\right]_{-1}^{1}$$

$$-\int_{-1}^{+1}\frac{d^{l-1}}{dx^l}(x^2-1)^{l-1}\frac{d^{l+1}(x^2-1)^l}{dx^{l+1}}dx$$

The first term on the RHS of above equation is zero as

$$\frac{d^l(x^2-1)^l}{dx^l} = l!\,(x+1)^l + l^2(x+1)^{l-1}\frac{l!}{2}(x-1) + l^2(l-1)^2$$

$$(x+1)^{l-2}\frac{l!}{3!}(x-1)^2 + ... + l^2(x-1)^{l-1}\frac{l!}{2}(x+1) + (x-1)^l\, l! = 0 \; for\; x = \pm 1$$

Thus the above integral could be rewritten as

$$\frac{1}{(2^l l!)^2}\int_{-1}^{+1}\frac{d^l(x^2-1)^l}{dx^l}\frac{d^l(x^2-1)^l}{dx^l}dx = -\frac{1}{(2^l l!)^2}\left[\int_{-1}^{1}\left(\frac{d^{l-1}(x^2-1)^l}{dx^{l-1}}\right)\left(\frac{d^{l+1}(x^2-1)^l}{dx^{l+1}}\right)dx\right]$$

Repeating the process l times, one can obtain

$$\frac{1}{(2^l l!)^2}\int_{-1}^{1}\left(\frac{d^l(x^2-1)^l}{dx^l}\right)\left(\frac{d^l(x^2-1)^l}{dx^l}\right)dx = \frac{(-1)^l}{(2^l l!)^2}\left[\int_{-1}^{1}(x^2-1)^l\left(\frac{d^{2l}(x^2-1)^l}{dx^{2l}}\right)dx\right] \quad ...(3.19)$$

Here $\left(\frac{d^{2l}(x^2-1)^l}{dx^{2l}}\right)$ can be solved using Libnitz formula as

$$\left(\frac{d^{2l}(x^2-1)^l}{dx^{2l}}\right) = (x+1)^l \frac{d^{2l}}{dx^{2l}}(x-1)^l + 2l\frac{d}{dx}(x+1)^l \frac{d^{2l-1}}{dx^{2l-1}}(x-1)^l + 2l(2l-1)\frac{d^2}{dx^2}(x+1)^l$$

$$\frac{d^{2l-2}}{dx^{2l-2}}(x-1)^l + ... + \frac{(2l)!}{l!l!}\frac{d}{dx}(x+1)^l \frac{d^{2l-1}}{dx^{2l-1}}(x-1)^l + ...$$

$$+ l\frac{d^{2l-1}}{dx^{2l-1}}(x+1)^l \frac{d}{dx}(x-1)^l + (x-1)^l \frac{d^{2l}}{dx^{2l}}(x+1)^l$$

or

$$\left(\frac{d^{2l}(x^2-1)^l}{dx^{2l}}\right) = \frac{(2l)!}{l!l!} l!l! = (2l)!$$

So that equation (3.19) becomes

$$\frac{1}{(2^l\, l!)^2}\int_{-1}^{1}\left(\frac{d^l(x^2-1)^l}{dx^l}\right)\left(\frac{d^l(x^2-1)^l}{dx^l}\right)dx = \frac{(-1)^l(2l)!}{(2^l\, l!)^2}\left[\int_{-1}^{1}(x^2-1)^l\, dx\right] \qquad ...(3.20)$$

The integral $\int_{-1}^{+1}(x^2-1)^l\, dx$ could be solved as

$$\int_{-1}^{1}(x^2-1)^l\, dx = \int_{-1}^{1}(x^2-1)^l(x-1)^l\, dx =$$

$$(x+1)^l . \frac{(x-1)^{l+1}}{l+1}\Bigg|_{-1}^{1} - \frac{l}{l+1}\int_{-1}^{1}(x+1)^{l-1}(x-1)^{l+1}\, dx$$

$$= -\frac{l}{l+1}\int_{-1}^{1}(x+1)^{l-1}(x-1)^{l+1}\, dx$$

Repeating the procedure *l* times, one will achieve the following result

$$\int_{-1}^{1}(x^2-1)^l\, dx = (-1)^l\frac{l!}{(l+1)(l+2)(l+3)...2l}\int_{-1}^{1}(x-1)^{2l}\, dx$$

$$= (-1)^l\frac{(l!)^2}{(2l)!}\frac{(x-1)^{2l+1}}{2l+1}\Bigg|_{-1}^{+1}$$

$$= (-1)^l\frac{(l!)^2\, 2^{2l+1}}{(2l+1)!} \qquad ...(3.21)$$

Substituting the value of integral in equation (3.20), the equation can be written as

$$\frac{1}{(2^l\, l!)^2}\int_{-1}^{1}\left(\frac{d^l\,(x^2-1)^l}{dx^l}\right)\left(\frac{d^l\,(x^2-1)^l}{dx^l}\right)dx = \frac{(-1)^l\,(2l)!}{(2^l\, l!)^2}\left[(-1)^l\,\frac{(l!)^2\,2^{2l+1}}{(2l+1)!}\right] = \frac{2}{2l+1}$$

Example 3.7. Show that

$$\int_{-1}^{1} x^l\,\mathrm{P}_m\,(x)\,dx = 0 \; for \; l < m \qquad \text{...(3.22)}$$

Solution :

$$\int_{-1}^{1} x^l\,\mathrm{P}_m\,(x)\,dx = \frac{1}{2^m\,m!}\int_{-1}^{1} x^l\left(\frac{d^m\,(x^2-1)^m}{dx^m}\right)dx$$

$$= \frac{1}{2^m\,m!}\left(x^l\left(\frac{d^{m-1}\,(x^2-1)^m}{dx^{m-1}}\right)\Bigg|_{-1}^{+1} - l\int_{-1}^{1} x^{l-1}\left(\frac{d^{m-1}\,(x^2-1)^m}{dx^{m-1}}\right)dx\right)$$

The first term on RHS of the above equation is zero due to the presence of $(x^2 - 1)^m$ in each term of the derivative, hence

$$\frac{1}{2^m\,m!}\int_{-1}^{1} x^l\left(\frac{d^m\,(x^2-1)^m}{dx^m}\right)dx = \frac{-l}{2^m\,m!}\int_{-1}^{1} x^{l-1}\left(\frac{d^{m-1}\,(x^2-1)^m}{dx^{m-1}}\right)dx$$

If the procedure is repeated $l + 1$ times, the term x^l will vanish and hence the integral will become zero for $l < m$.

Example 3.8. Show that

$$\int_{-1}^{1} x^l\,\mathrm{P}_l\,(x)\,dx = \frac{2^{2l+1}\,(l!)^2}{(2l+1)!} \qquad \text{...(3.23)}$$

Solution :

$$\int_{-1}^{1} x^l\,\mathrm{P}_l\,(x)\,dx = \frac{1}{2^l\,l!}\int_{-1}^{1} x^l\left(\frac{d^l\,(x^2-1)^l}{dx^l}\right)dx$$

$$= \frac{1}{2^l\,l!}\left\{x^l\left(\frac{d^{l-1}\,(x^2-1)^l}{dx^{l-1}}\right)\Bigg|_{-1}^{+1} - l\int_{-1}^{1} x^{l-1}\left(\frac{d^{l-1}\,(x^2-1)^l}{dx^{l-1}}\right)dx\right\}$$

The first term on RHS of the above equation is zero due to the presence of $(x^2 - 1)^l$ in each term of the derivative, hence

$$\frac{1}{2^l\,l!}\int_{-1}^{1} x^l\left(\frac{d^l\,(x^2-1)^l}{dx^l}\right)dx = \frac{-l}{2^l\,l!}\int_{-1}^{1} x^{l-1}\left(\frac{d^{l-1}\,(x^2-1)^l}{dx^{l-1}}\right)dx$$

Repeating the procedure l times, one can achieve the result

$$\frac{1}{2^l\, l!}\int_{-1}^{1} x^l \left(\frac{d^l\,(x^2-1)^l}{dx^l}\right) dx = \frac{(-1)^l\, l!}{2^l\, l!}\int_{-1}^{1}(x^2-1)^l\, dx$$

Using the results of equation (3.20), the above integral becomes

$$\frac{1}{2^l\, l!}\int_{-1}^{1} x^l \left(\frac{d^l\,(x^2-1)^l}{dx^l}\right) dx = \frac{(l!)^2\, 2^{l+1}}{(2l+1)!}$$

Example 3.9. Evaluate

$$\int_{-1}^{1} x\mathrm{P}_l\,(x)\,\mathrm{P}_m\,(x)\,dx$$

Solution : From equation (3.10), one may have

$$\frac{1}{(2l+1)}\,[(l+1)\,\mathrm{P}_{l+1}\,(x) + l\,\mathrm{P}_{l-1}\,(x)] = x\mathrm{P}_l\,(x)$$

So that the equation becomes

$$\int_{-1}^{1} x\mathrm{P}_l\,(x)\mathrm{P}_m\,(x)\,dx = \frac{1}{(2l+1)}\int_{-1}^{1}[(l+1)\mathrm{P}_{l+1}\,(x)+l\,\mathrm{P}_{l-1}\,(x)]\,\mathrm{P}_m\,(x)\,dx$$

$$= \frac{1}{(2l+1)}\left[(l+1)\int_{-1}^{1}\mathrm{P}_{l+1}\,(x)\mathrm{P}_m\,(x)\,dx + l\int_{-1}^{1}\mathrm{P}_{l-1}\,(x)\mathrm{P}_m\,(x)\,dx\right]$$

The integrals will become zero unless $m = l \pm 1$, thus

$$\int_{-1}^{1} x\mathrm{P}_l\,(x)\mathrm{P}_m\,(x)\,dx = \frac{1}{(2l+1)}\left[(l+1)\frac{2}{2(l+1)+1} + l\frac{2}{2(l-1)+1}\right]$$

$$= \frac{2}{(2l+1)}\left(\frac{l+1}{2l+3} + \frac{2l}{2l-1}\right)$$

Example 3.10. Prove that the Legendre's Polynomials can be represented by the definite integral

$$\mathbf{P}_l\,(x) = \frac{1}{\pi}\int_0^{\pi}[x \mp \sqrt{(x^2-1)}\ \cos\phi]^l\, d\phi \qquad \textbf{...(3.24)}$$

Solution : To start with consider the defiinite integral

$$\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}} \quad (\text{if } a > b) \qquad ...(3.25)$$

Let $a = 1 - xt$ and $b = t\sqrt{(x^2 - 1)}$

so that
$$a^2 - b^2 = (1 - xt)^2 - t^2(x^2 - 1)$$
$$= 1 - 2xt + x^2t^2 - t^2x^2 + t^2$$
$$= 1 - 2xt + t^2$$

so that the integral given by equation (3.23) could be modified as

$$\int_0^{\pi} \frac{d\phi}{(1 - xt) \pm t\sqrt{(x^2 - 1)}\ \cos\phi} = \frac{\pi}{\sqrt{1 - 2xt + t^2}}$$

$$\pi(1 - 2xt + t^2)^{-1/2} = \int_0^{\pi} \frac{d\phi}{(1 - xt) \pm t\sqrt{(x^2 - 1)}\cos\phi}$$

As $(1 - 2xt + t^2)^{-1/2}$ is the generating function of Legendre's polynomials hence using equation (3.8) one gets

$$\pi \sum P_l(x)t^l = \int_0^{\pi} \frac{d\phi}{1 - t[x \pm \sqrt{(x^2 - 1)}\ \cos\phi]}$$

$$= \int_0^{\pi} [1 - t\{x \pm \sqrt{(x^2 - 1)}\cos\phi\}^{-1}\ d\phi$$

$$\pi \sum P_l(x)t^l = \int_0^{\pi} (1 - z)^{-1} d\phi \qquad \text{...(3.26)}$$

Here $z = t\{x \mp \sqrt{(x^2 - 1)}\ \cos\phi\}$

Eqn. (3.26) could be rewritten as

$$\pi \sum_{l=0}^{\infty} P_l(x)t^l = \int_0^{\pi} (1 + z + z^2 + ...z^l)d\phi$$

$$= \sum_{l=0}^{\infty} \int_0^{\pi} z^l\ d\phi$$

$$= \sum_{l=0}^{\infty} \int_0^{\pi} t^l\ (x \mp \sqrt{(x^2 - 1)}\ \cos\phi)^l\ d\phi$$

Comparing the coefficients of t^l on both sides, one gets

$$\pi P_l(x) = \int_0^{\pi} \{x \mp \sqrt{(x^2 - 1)}\ \cos\phi\}^l\ d\phi$$

or $$P_l(x) = \frac{1}{\pi}\int_0^{\pi}\{x \mp \sqrt{(x^2-1)}\,\cos\phi\}^l\,d\phi$$

This integral is also known as Laplace's first definite integral.

Example 3.11. Evaluate $\int_{-1}^{+1} x^2 P_l^2(x)\,dx$

Solution : To start with consider the recurrence relation

$$(l+1)\,P_{l+1}(x) = (2l+1)\,x\,P_l(x) - l\,P_{l-1}(x)$$

or $$(2l+1)x\,P_l(x) = (l+1)\,P_{l+1}(x) + l\,P_{l-1}(x)$$

Squaring both sides and integrating both sides with respect to x between the limits –1 to + 1, one gets

$$(2l+1)^2\int_{-1}^{+1} x^2 P_l^2(x)\,dx = (l+1)^2\int_{-1}^{+1} P_{l+1}^2(x)\,dx + l^2\int_{-1}^{+1} P_{l-1}^2(x)\,dx + 2l(l+1)\int_{-1}^{+1} P_{l+1}(x)\,P_{l-1}(x)\,dx$$

Using the orthogonality relation of Legendre's polynomials given by equation (3.16) one gets

$$(2l+1)^2\int_{-1}^{+1} x^2 P_l^2(x)\,dx = (l+1)^2\frac{2}{2(l+1)+1} + l^2\frac{2}{2(l-1)+1} + 0$$

$$= \frac{2(l+1)^2}{2l+3} + \frac{2l^2}{2l-1}$$

or $$\int_{-1}^{+1} x^2 P_l^2(x)\,dx = \frac{2(l+1)^2}{(2l+1)^2\,(2l+3)} + \frac{2l^2}{(2l+1)^2\,(2l-1)}$$

Example 3.12. Find the value of l if

$$\int_{-1}^{+1} P_l(x)\,dx = 2$$

Solution : From the result of example (3.23) one get

$$\int_{-1}^{+1} x^l P_l(x)\,dx = \frac{2^{2l+1}(l!)^2}{(2l+1)!}$$

which will be 2 iff $l = 0$ as

$$\frac{2^{2l+1}(l!)^2}{(2l+1)!} = \frac{2^1(0!)^2}{1!} = 2$$

Hence 0 is the value of l for

$$\int_{-1}^{+1} P_l(x)\,dx = 2$$

Example 13. Find the value of $P_{2l+1}(0)$.

Solution : Consider the generating function of Legendre's polynomials

$$\sum_{l=0}^{\infty} P_{2l}(x)\,t^{2l} = (1 - 2xt + t^2)^{-1/2}$$

substituting $x = 0$, one gets

$$\sum_{l=0}^{\infty} P_{2l}(0)\,t^{2l} = (1 + t^2)^{-1/2}$$

or

$$\sum_{l=0}^{\infty} P_{2l}(0)\,t^{2l} = 1 - \frac{1}{2}t^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}(t^2)^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(t^2)^3 + \ldots$$

$$+ \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(t^2)^n$$

Comparing the coefficients of t^{2l+1} on both sides one gets

$$P_{2l+1}(0) = 0$$

3.6. TRIGNOMETRIC REPRESENTATION OF LEGENDRE'S POLYNOMIALS

The Legendre's polynomials can also be written in terms of trigonometric functions by substituting $x = \cos\theta$, such that equation (3.5) becomes

$$P_l(\cos\theta) = \frac{1}{2^l\, l!}\left(\frac{d^l(\cos^2\theta - 1)^l}{d\theta^l}\cdot\left(\frac{dx}{d\theta}\right)^l\right) = \frac{1}{2^l\, l!}\left((-1)^l\,\frac{d^l \sin^{2l}\theta}{d\theta^l}\cdot\left(\frac{-1}{\sin\theta}\right)^l\right)$$

$$= \frac{1}{2^l\, l!}\left(\frac{1}{\sin^l\theta}\,\frac{d^l \sin^{2l}\theta}{d\theta^l}\right) \qquad \ldots(3.27)$$

3.7. SECOND KIND OF LEGENDRE'S POLYNOMIALS $Q_l(x)$

The second solution to Legendre's differential equation, given by the second part of the equation (3.3)

$$y = a_0' \sum_{n=0}^{\infty} \frac{(l+1)(l+2)(l+3)(l+4)\ldots(l-2n)}{2^n\, n!(2l+3)(2l+5)\ldots(2l+2n+1)}\,x^{-l-2n-1}$$

is known as second kind of Legendre's polynomials and are denoted by $Q_l(x)$. The relation between $P_l(x)$ and $Q_l(x)$ is given as

$$Q_l(x) = \frac{1}{2} P_l(x) \ln \frac{1+x}{1-x}$$

with $$Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} \text{ as } P_1(x) = 1$$

and $$Q_1(x) = \frac{1}{2} P_1(x) \ln \frac{1+x}{1-x} = P_1(x) Q_0(x) - 1$$

3.8. ASSOCIATED LEGENDRE'S POLYNOMIALS

The solution to associated Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(l(l+1) - \frac{m}{1-x^2} \right) y = 0$$

And are given as

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \left(\frac{d^m}{dx^m} P_l(x) \right), \; m \geq 0 \qquad \text{...(3.28)}$$

That can be extended for negative values of *m* by finding the proportionality between $P_l^m(x)$ and $P_l^{-m}(x)$ by substituting the value of $P_l(x)$ from equation (3.5), so that equation (3.28) can be rewritten as

$$P_l^m(x) = \frac{(-1)^m (1-x^2)^{m/2}}{2^l \, l!} \left(\frac{d^{l+m}}{dx^{l+m}} (x-1)^l (x+1)^l \right) \qquad \text{...(3.29)}$$

Differentiating equation (3.29) using Libnitz formula one gets

$$P_l^m(x) = \frac{(-1)^m (1-x^2)^{m/2}}{2^l \, l!} \left((x+1)^{l+m} \frac{d^{l+m}}{dx^{l+m}} (x-1)^l + (l+m) \frac{d}{dx}(x+1)^l \frac{d^{l+m-1}}{dx^{l+m-1}} (x-1)^l + \right.$$

$$\ldots + {}^{l+m}C_m \frac{d^m}{dx^m}(x+1)^l + \ldots + {}^{l+m}C_l \frac{d^l}{dx^l}(x+1)^l \frac{d^m}{dx^m}(x-1)^l + \ldots$$

$$\left. + (l+m) \frac{d^{l+m-1}}{dx^{l+m-1}}(x+1)^l \frac{d}{dx}(x-1)^l + (x-1)^l \frac{d^{l+m}}{dx^{l+m}}(x+1)^l \right)$$

In the above expression, the terms varying between *m* to l are non zero, therefore the above term can be rewritten as

$$P_l^m(x) = \frac{(-1)^m (1-x^2)^{m/2}}{2^l \, l!} \sum_{r=m}^{l} \frac{(l+m)!}{r!(l+m-r)!} \frac{l!}{(l-r)!} (x+1)^{l-r} \frac{l!}{(r-m)!} (x-1)^{r-m}$$

Replacing $r' = r - m$, the above equation could be rewritten as

$$P_l^m(x) = \frac{(-1)^m (1-x^2)^{m/2}}{2^l \, l!} \sum_{r'=0}^{l-m} \frac{(l+m)!}{(r'+m)!(l-r')!} \frac{l!}{(l-r'-m)!} (x+1)^{l-r'-m} \frac{l!}{(r')!} (x-1) r'$$

Multiplying and dividing the above equation with $(1 - x^2)^{-m}$ $(l - m)!$ And rearranging the terms, one gets

$$P_l^m(x) = \frac{(-1)^m (1-x^2)^{-\frac{m}{2}}}{2^l\, l!}(-1)^m (x^2-1)^{-m}\frac{(l+m)!}{(l-m)!}$$

$$\sum_{r'=0}^{l-m}\frac{(l-m)!}{(r')!(l-r'-m)!}\frac{l!}{(l-r')!}(x+1)^{l-r'-m}\frac{l!}{(r'+m)!}(x-1)r'$$

$$\text{Or } P_l^m(x) = \frac{(-1)^m (1-x^2)^{-\frac{m}{2}}}{2^l\, l!}(-1)^m\frac{(l+m)!}{(l-m)!}\sum_{r'=0}^{l-m}\frac{(l-m)!}{(r')!(l-r'-m)!}\frac{l!}{(l-r')!}(x+1)^{l-r'}$$

$$\frac{l!}{(r'+m)!}(x-1)^{r'+m}$$

$$\text{Or } P_l^m(x) = \frac{(-1)^m (1-x^2)^{-\frac{m}{2}}}{2^l\, l!}(-1)^m\frac{(l+m)!}{(l-m)!}\sum_{r'=0}^{l-m}{}^{(l-m)}C_{r'}\frac{d^{r'}}{dx^{r'}}(x+1)^l\frac{d^{l-m-r'}}{dx^{l-m-r'}}(x-1)^l$$

$$\text{Or } P_l^m(x) = (-1)^m\frac{(l+m)!}{(l-m)!}(-1)^m\frac{(1-x^2)^{-\frac{m}{2}}}{2^l\, l!}\sum_{r'=0}^{l-m}{}^{(l-m)}C_{r'}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l$$

$$\text{Or } P_l^m(x) = (-1)^m\frac{(l+m)!}{(l-m)!}P_l^{-m}(x) \qquad ...(3.30)$$

3.8.1. Recurrence relations of Associated Legendre's Polynomials

Recurrence relations of associated Legendre's polynomials can be obtained directly from the recurrence relations of Legendre's polynomials. The equation has its role in solving spherical harmonics utilized to quantum mechanics to solve the θ and ϕ part of spherically symmetric problems.

The first recurrence relation can be obtained by differentiating equation (3.10) m times with respect to x one gets

$$(l+1)\frac{d^m P_{l+1}(x)}{dx^m} = (2l+1)\frac{d^m(xP_l(x))}{dx^m} - l\frac{d^m P_{l-1}(x)}{dx^m}$$

$$\text{Or} \quad (l+1)\frac{d^m P_{l+1}(x)}{dx^m} = (2l+1)x\frac{d^m P_l(x)}{dx^m} + (2l+1)m\frac{d^m P_l(x)}{dx^m} - l\frac{d^m P_{l-1}(x)}{dx^m}$$

Multiplying the above equation by the factor of $(-1)^m\,(1-x^2)^{\frac{m}{2}}$, one gets

$$(l+1)(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m P_{l+1}(x)}{dx^m} = (2l+1)\,x\,(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m P_l(x)}{dx^m} + (2l+1)\,m\,(-1)^m$$

$$(1-x^2)^{\frac{m}{2}}\frac{d^m P_l(x)}{dx^m} - l(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m P_{l-1}(x)}{dx^m}$$

Using the definition of associated Legendre's polynomials, the above equation could be rewritten as

$$\mathbf{(l+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x)+(2l+1)m(1-x^2)^{\frac{1}{2}}P_l^{m-1}(x)-lP_{l-1}^m(x)}$$

...(3.31)

The other recurrence relation can be obtained by differentiating equation (3.13) m-1 times with respect to x and multiplying the result by a factor of $(-1)^m(1-x^2)^{\frac{m}{2}}$

$$(-1)^m(1-x^2)^{m/2}\frac{d^{m-1}}{dx^{m-1}}(xP_l'(x)) - (-1)^m(1-x^2)^{m/2}\frac{d^m P_l(x)}{d_x{}^m}$$

$$= l(-1)^m(1-x^2)^{m/2}\frac{d^{m-1}}{dx^{m-1}}P_l(x)$$

$$l(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^{m-1}P_l(x)}{dx^{m-1}} = x(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m P_l(x)}{dx^m}+m(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^{m-1}P_l(x)}{dx^{m-1}}$$

$$-(-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m P_{l-1}(x)}{dx^m}$$

Using the definition of associated Legendre's polynomials, the above equation could be rewritten as

$$-(1-x^2)^{\frac{1}{2}}\,lP_l^{m-1}(x) = xP_l^m(x)-(m-1)(1-x^2)^{\frac{1}{2}}P_l^{m-1}(x)+P_{l-1}^m(x)$$

Or $\mathbf{[P_{l-1}^m(x)-(l-m+1)(1-x^2)^{\frac{1}{2}}P_l^{m-1}(x) = xP_l^m(x)]}$ ...(3.32)

The third recurrence relation can be obtained by eliminating $(1-x^2)^{\frac{1}{2}}P_l^m(x)$ from equation (3.28) and (3.32)

$\mathbf{(l-m+1)P_{l+1}^m(x)-(2l+1)\,xP_l^m(x)+(l+m)P_{l-1}^m(x)}$ ...(3.33)

3.8.2. Orthogonality of Associated Legendre's Polynomials

The orthonormality of $P_l^m(x)$ and $P_{l'}^m(x)$ for $l' \neq l$ can be obtained using the Rodrigue's formula for associated Legendre's differential equation as

$$\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l'}^m(x)dx = (-1)^m \frac{(l+m)!}{(l-m)!}\int_{-1}^{1} \mathrm{P}_l^{-m}(x)\mathrm{P}_{l'}^m(x)dx \qquad ...(3.34)$$

Here the result of equation (3.30) has been utilized. Equation (3.29) could be utilized in the above equation as

$$= \frac{(-1)^{3m}(l+m)!}{2^l\, l!2^{l'}\, l'!(l-m)!}\int_{-1}^{1}\left((1-x^2)^{-m/2}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l\right)\left((1-x^2)^{m/2}\frac{d^{l'+m}}{dx^{l'+m}}(x^2-1)^{l'}\right)dx$$

Or $$\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l'}^m(x)dx = \frac{(-1)^{3m}(l+m)!}{2^l\, l!2^{l'}\, l'!(l-m)!}\int_{-1}^{1}\left(\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l\right)\left(\frac{d^{l'+m}}{dx^{l'+m}}(x^2-1)^{l'}\right)dx$$

Integrating the above equation by parts, one gets

$$\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l'}^m(x)dx = \frac{(-1)^{3m}}{2^l\, l!2^{l'}\, l'!}\frac{(l+m)}{(1-m)!}\left[\left(\frac{d^{l'+m-1}}{dx^{l'+m-1}}(x^2-1)^l\right)\left(\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^{l'}\right)\Bigg|_{-1}^{1}\right.$$

$$\left. - \int_{-1}^{1}\left(\frac{d^{l-m+1}}{dx^{l-m+1}}(x^2-1)^l\right)\left(\frac{d^{l'+m-1}}{dx^{l'+m-1}}(x^2-1)^{l'}\right)dx\right]$$

The first term on the right hand side will be zero as all the differential terms contain (x^2 – 1) that will vanish for $x = \pm 1$, so that the above equation could be rewritten as

$$\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l'}^m(x)dx = \frac{(-1)^{3m+1}}{2^l\, l!2^{l'}\, l'!}\frac{(l+m)!}{(l-m)!}\left[\int_{-1}^{1}\left(\frac{d^{l-m+1}}{dx^{l-m+1}}(x^2-1)^l\right)\left(\frac{d^{l'+m-1}}{dx^{l'+m-1}}(x^2-1)^{l'}\right)dx\right]$$

Repeating the same process *m* times, one gets

$$\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l'}^m(x)dx = \frac{(-1)^{4m}}{2^l\, l!2^{l'}\, l'!}\frac{(l+m)!}{(l-m)!}\left[\int_{-1}^{1}\left(\frac{d^{l}}{dx^{l}}(x^2-1)^l\right)\left(\frac{d^{l'}}{dx^{l'}}(x^2-1)^{l'}\right)dx\right]$$

Using the Rodrigue's definition of $\mathrm{P}_l(x)$, the above equation could be rewritten as

$$\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l,}^m(x)dx = \frac{(l+m)!}{(l-m)!}\left[\int_{-1}^{1} \mathrm{P}_l(x)\mathrm{P}_{l,}(x)dx\right]$$

Using the results of equation (3.16), one is able to rewrite the above equation as

$$\boldsymbol{\int_{-1}^{1} \mathrm{P}_l^m(x)\mathrm{P}_{l'}^m(x)dx = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{lm}} \qquad ...(3.35)$$

That is the required orthogonality relation.

4. HERMITE'S DIFFERENTIAL EQUATION AND IT'S SOLUTION

The Hermite's differential equation is given as

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2vy = 0 \qquad ...(4.1)$$

The equation is utilized in quantum mechanics to solve the problem of simple harmonic oscillator and to find its eigen values and eigen functions. It is also used in molecular spectroscopy to find the vibrational spectrum of the molecules, thus keeping in view the importance of the equation, it is necessary to solve this differential equation. The equation (4.1) is already in its general form with P $(x) = -2x$ and $Q(x) = 2v$ finite for all real numbers, hence no singular point exists for the differential equation (4.1). The general series solution to the equation is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

such that

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

And

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the values of y, y', y'' in equation (4.1) one can obtain the relation

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + 2v\sum_{n=0}^{\infty} a_n x^n = 0$$

Or $$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=0}^{\infty} n a_n x^{n} + 2v\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=0}^{\infty} (n-v) a_n x^{n} = 0$$

Equating the coefficients of x^n on both sides one can obtain

$$(n+2)(n+1)\, a_{n+2} - 2(n-v)\, a_n = 0$$

Such that

$$a_{n+2} = -\frac{2(v-n)}{(n+2)(n+1)} a_n$$

Substituting $n = 0, 1, 2, 3, 4$ ...., one can obtain the values of $a_n's$ in terms of a_0 and a_1 as

$$a_2 = -\frac{2v}{2.1} a_0, \qquad a_3 = -\frac{2(v-1)}{3.2} a_1,$$

$$a_4 = -\frac{2(v-2)}{4.3} a_2 = (-1)^2 \frac{2^2 v(v-2)}{4!} a_0 \qquad a_5 = -\frac{2(v-3)}{5.4} a_3 = (-1)^2 \frac{2^2 (v-1)(v-3)}{5!} a_1,$$

$$a_6 = -\frac{2(v-4)}{6.5}a_4 = (-1)^3\frac{2^3 v(v-2)(v-4)}{6!}a_0 \qquad a_7 = -\frac{2(v-5)}{7.6}a_5 = (-1)^3\frac{2^3 v(v-1)(v-3)(v-5)}{7!}a_1$$

$$a_{2n} = \frac{(-2)^n v(v-2)(v-4)...(v-2n-2)}{2n!}a_0 \qquad a_{2n+1} = \frac{(-2)^n (v-1)(v-3)(v-5)...(v-2n+1)}{(2n+1)!}a_1$$

Thus the solution to equation (4.1) is

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-2)^n v(v-2)(v-4)...(v-2n-2)}{2n!} x^{2n}$$

$$+a_1 \sum_{n=0}^{\infty} \frac{(-2)^n (v-1)(v-3)(v-5)...(v-2n+1)}{(2n+1)!} x^{2n+1}$$

Additional Information (Taylor Series)

Taylor's series have been designated so honouring Brook Taylor who invented these series in 1715 and is a series expansion of a function about a regular point and is given as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ ... + \frac{(x-a)^n}{n!}f^n(a)$$

Where a is a regular point and $f^n(a)$ is the nth derivative of $f(x)$ at $x = a$

4.1. GENERATING FUNCTIONS OF HERMITE POLYNOMIAL

The solutions to Hermite's Differential equation are known as Hermite Polynomials and the generating function of Hermite's polynomials is

$$e^{2xt-t^2} = \sum_{v=0}^{\infty} \frac{\mathbf{H}_v(x)t^v}{v!} \qquad ...(4.2)$$

To verify it consider the Taylor series expansion of e^{2xt-t^2} around $t = 0$

$$e^{2xt-t^2} = 1 + t\frac{\partial}{\partial t}e^{2xt-t^2} + \frac{t^2}{2!}\frac{\partial^2}{\partial t^2}e^{2xt-t^2} + \frac{t^3}{3!}\frac{\partial^3}{\partial t^3}e^{2xt-t^2} + ...$$

Or

$$e^{2xt-t^2} = 1 + te^{x^2}\frac{\partial}{\partial t}e^{-(x^2-2xt+t^2)} + \frac{e^{x^2}t^2}{2!}\frac{\partial^2}{\partial t^2}e^{-(x^2-2xt+t^2)}$$

$$+\frac{e^{x^2}t^3}{3!}\frac{\partial^3}{\partial t^3}e^{-(x^2-2xt-t^2)} + ...$$

Or

$$e^{2xt-t^2} = 1 + te^{x^2}\frac{\partial}{\partial t}e^{-(x-t)^2} + \frac{e^{x^2}t^2}{2!}\frac{\partial^2}{\partial t^2}e^{-(x-t)^2} + \frac{e^{x^2}t^3}{3!}\frac{\partial^3}{\partial t^3}e^{-(x-t)^2} + ...$$

...(4.3)

Since $$\frac{\partial}{\partial t}\mathrm{F}(x-t) = -\frac{\partial}{\partial x}\mathrm{F}(x-t)$$

Equation (4.3) can be rewritten as

$$g\,(x,t) = e^{2xt-t^2} = 1 - te^{x^2}\frac{\partial}{\partial x}e^{-(x-t)^2} + \frac{e^{x^2}t^2}{2!}\frac{\partial^2}{\partial xx^2}e^{-(x-t)^2} - \frac{e^{x^2}t^3}{3!}\frac{\partial^3}{\partial x^3}e^{-(x-t)^2} + ...$$

$$= \sum_{v=0}^{\infty}(-1)^v\frac{e^{x^2}t^v}{v!}\frac{\partial^v}{\partial x^v}e^{-(x-t)^2}$$

At $t = 0$, the above equation becomes

$$g\,(x) = \sum_{v=0}^{\infty}(-1)^v\frac{e^{x^2}}{v!}\frac{\partial^v}{\partial x^v}e^{-x^2} = \sum_{v=0}^{\infty}\frac{f_v(x)}{v!}$$

Where

$$f_v\,(x) = (-1)^v e^{x^2}\frac{\partial^v}{\partial x^v}e^{-x^2} \quad ...(4.4)$$

To ensure if $f_v\,(x)$ is the solution to eqn (4.1) consider $\mathrm{F} = e^{-x^2}$ and its differentiation with respect to x, such that

$$\frac{d\,\mathrm{F}}{dx} = -2xe^{-x^2} = -2xF \quad ...(4.5)$$

Differentiate equation (4.5) $v + 1$ times using Libnitz formula, one will obtain

$$\frac{d^{v+2}\mathrm{F}}{dx^{v+2}} = -2x\frac{d^{v+1}\mathrm{F}}{dx^{v+1}} - 2v\frac{d^v\mathrm{F}}{dx^v}$$

Or $$\frac{d^{v+2}\mathrm{F}}{dx^{v+2}} = -2x\frac{d^{v+1}\mathrm{F}}{dx^{v+1}} - 2v\frac{d^v\mathrm{F}}{dx^v}$$

Or $$\frac{d^2}{dx^2}\left(\frac{d^vF}{dx^v}\right) = -2x\frac{d}{dx}\left(\frac{d^v\mathrm{F}}{dx}\right) - 2v\frac{d^v\mathrm{F}}{dx^v} \quad ...(4.6)$$

From equation (4.4) one may obtain

$$\frac{d^v\mathrm{F}}{dx^v} = \frac{d^ve^{-x^2}}{dx^v} = (-1)^v e^{-x^2} f_v\,(x)$$

Substituting the value in equation (4.6) one obtains

$$\frac{d^2}{dx^2}((-1)^v e^{-x^2} f_v(x)) = -2x\frac{d}{dx}((-1)^v e^{-x^2} f_v(x)) - 2v(-1)^v e^{-x^2} f_v(x)$$

Or $$(-1)^{\nu}\frac{d^2}{dx^2}(e^{-x^2}f_{\nu}(x)) = -2x(-1)^{\nu}\frac{d}{dx}(e^{-x^2}f_{\nu}(x))-2\nu(-1)^{\nu}e^{-x^2}f_{\nu}(x)$$

Or $$(-1)^{\nu}\left(e^{-x^2}\frac{d^2f_{\nu}(x)}{dx^2}-4xe^{-x^2}\frac{df_{\nu}(x)}{dx}+4x^2e^{-x^2}f_{\nu}(x)\right)$$

$$= -2x(-1)^{\nu}\left(e^{-x^2}\frac{df_{\nu}(x)}{dx}-2xe^{-x^2}f_{\nu}(x)\right)-2\nu(-1)^{\nu}e^{-x^2}f_{\nu}(x)$$

Or $$(-1)^{\nu}\left(e^{-x^2}\frac{d^2f_{\nu}(x)}{dx^2}-2xe^{-x^2}\frac{df_{\nu}(x)}{dx}\right) = -2\nu(-1)^{\nu}e^{-x^2}f_{\nu}(x) \quad ...(4.7)$$

Eliminating $(-1)^{\nu}e^{-x^2}$ from equation (4.7), one can get

$$\frac{d^2f_{\nu}(x)}{dx^2}-2x\frac{df_{\nu}(x)}{dx}+2\nu f_{\nu}(x) = 0$$

Therefore, $f_{\nu}(x) = H_{\nu}(x)$ or

$$\mathbf{H_{\nu}(x) = (-1)^{\nu}e^{x^2}\frac{\partial^{\nu}}{\partial x^{\nu}}e^{-x^2}} \quad ...(4.8)$$

This is the solution to equation (4.7) and is also known as Rodrigue's formula for Hermite polynomials.

Example 4.1. **Prove that $H_{\nu}(-x) = (-1)^{\nu}H_{\nu}(x)$.**

Solution : Replace x with $-x$ in the generating function of Hermite's polynomials given in equation (4.2), one gets

$$e^{-2xt-t^2} = \sum_{\nu=0}^{\infty}\frac{H_{\nu}(-x)t^{\nu}}{\nu!}$$

Or $$e^{2x(-t)-(-t)^2} = \sum_{\nu=0}^{\infty}\frac{H_{\nu}(-x)t^{\nu}}{\nu!}$$

Or $$\sum_{\nu=0}^{\infty}\frac{H_{\nu}(x)(-t)^{\nu}}{\nu!} = \sum_{\nu=0}^{\infty}\frac{H_{\nu}(-x)t^{\nu}}{\nu!}$$

Comparing the coefficients of t^{ν} on both sides, one gets

$$H_{\nu}(-x) = (-1)^{\nu}H_{\nu}(x)$$

Example 4.2. **Convert $2H_4(x) + 3H_4(x) - H_2(x) + 5H_1(x) + 6H_0(x)$ into ordinary polynomial.**

Solution : Here

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_3(x) = 8x^3 - 12x$$
$$H_2(x) = 4x^2 - 2$$
$$H_1(x) = 2x$$
$$H_0(x) = 1$$

Substituting the values of Hermite's polynomials in the given equation one can obtains

$2(16x^4 - 48x^2 + 12) + 3(8x^3 - 12x) - (4x^2 - 2) + 5.2x + 6.1$

$32x^4 + 24x^3 - 100x^2 - 26x + 32$

4.2. ORTHOGONALITY OF HERMITE'S POLYNOMIALS

The orthogonalityof Hermite's polynomials is given as

$$\int_{-\infty}^{\infty} e^{-x^2} \mathbf{H}_v(x)\, \mathbf{H}_{v'}(x)dx = \sqrt{\pi}\, 2^v . v! \delta_{vv'}$$

And can be deduced by utilizing the generating function

$$e^{2xt-t^2} e^{2xs-s^2} = \sum_{v'=0}^{\infty} \sum_{v'=0}^{\infty} H_v(x) H_{v'}(x) \frac{t^v}{v!} \frac{s^{v'}}{v'!}$$

Multiplying both sides by e^{-x^2} and integrating between the limits $-\infty$ to ∞, one gets

$$\int_{-\infty}^{\infty} e^{-x^2}\, H_v(x)\, H_{v'}(x)dx \frac{t^v}{v!} \frac{s^{v'}}{v'!} = \int_{-\infty}^{\infty} e^{-x^2}\, e^{2xt-t^2} e^{2xs-s^2} dx$$

Or $$\int_{-\infty}^{\infty} e^{-x^2}\, H_v(x)\, H_{v'}(x)dx \frac{t^v}{v!} \frac{s^{v'}}{v'!} = \int_{-\infty}^{\infty} e^{-x^2+2x(t+s)-t^2-s^2} dx \qquad ...(4.9)$$

Multiplying and dividing RHS of equation (4.9) with $e^{2st,}$ the following result can be achieved

$$\int_{-\infty}^{\infty} e^{-x^2} H_v(x) H_{v'}(x)\, dx \frac{t^v}{v!} \frac{s^{v'}}{v'!} = e^{-2st} \int_{-\infty}^{\infty} e^{-(x-t-s)^2} dx = e^{-2st}\sqrt{\pi}$$

As $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$ is called the integral of Gaussian.

$$\int_{-\infty}^{\infty} e^{-x^2} H_v(x)\, H_{v'}(x)\, dx \frac{t^v}{v!} \frac{s^{v'}}{v'!} = e^{-2st}\sqrt{\pi} \qquad ...(4.10)$$

Here $$e^{-2st} = 1 - 2st + \frac{4s^2t^2}{2!} - .... = \sum_{v=0}^{\infty} \frac{2^{v'} s^{v'} t^{v'}}{v'!}$$

Thus equation (4.10) can be rewritten as

$$\int_{-\infty}^{\infty} e^{-x^2} H_v(x) H_{v'}(x)\, dx \frac{t^v}{v!} \frac{s^{v'}}{v'!} = \sqrt{\pi} \sum_{v=0}^{\infty} \frac{2^{v'} s^{v'} t^{v'}}{v'!}$$

Comparing the coefficients of $t^v s^{v'}$ on both sides, it could be concluded that

$$\int_{-\infty}^{\infty} e^{-x^2} \mathbf{H}_v(x)\, \mathbf{H}_{v'}(x)\, dx = \begin{cases} 2^v v! \sqrt{\pi} & \text{if } v' = v \\ 0 & \text{if } v' \neq v \end{cases}$$

The orthogonality of Hermite's polynomials can be used to expand any arbitrary function $f(x)$ in terms of Hermite's polynomials as

$$f(x) = \sum_{v=0}^{\infty} a_v \mathrm{H}_v(x) \qquad \text{...(4.11)}$$

The coefficient a_v can be found by multiplying equation (4.11) with $e^{-x^2}\, \mathrm{H}_{v'}(x)$ and integrating both sides between the limits $-\infty$ to ∞,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, \mathrm{H}_{v'}(x)\, dx = \sum_{v=0}^{\infty} a_v \int_{-\infty}^{\infty} e^{-x^2} \mathrm{H}_v(x)\, \mathrm{H}_{v'}(x)\, dx$$

Or
$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, \mathrm{H}_{v'}(x)\, dx = \sum_{v=0}^{\infty} a_v 2^v v! \sqrt{\pi}\, \delta_{vv'}$$

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, \mathrm{H}_{v'}(x)\, dx = a_{v'}\, 2^{v'} v'! \sqrt{\pi}$$

Or
$$\frac{1}{2^{v'} v'! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) \mathrm{H}_{v'}(x)\, dx = a_{v'} \qquad \text{...(4.12)}$$

Example 4.3. Express $f(x) = e^{2bx}$ in terms of Hermite polynomials and use the result to deduce the value of integral

$$\int_{-\infty}^{\infty} e^{-x^2+2bx} \mathrm{H}_v(x)\, dx$$

Solution : Consider generating function equation (4.2) of Hermite's polynomials and replace t with b

$$e^{2xb-b^2} = \sum_{n=0}^{\infty} \frac{\mathrm{H}'_v(x)\, b^v}{v!}$$

Or
$$e^{2xb} = e^{b^2} \sum_{v=0}^{\infty} \frac{\mathrm{H}'_v(x)\, b^v}{v!}$$

But $f(x) = e^{2bx}$ hence

$$f(x) = e^{b^2} \sum_{v=0}^{\infty} \frac{\mathrm{H}'_v(x)\, b^v}{v!} \qquad \text{...(4.13)}$$

Comparing the results obtained in equation (4.13) with equation (4.11), one can obtain

$$a_v = \frac{b^v}{v!}e^{b^2}$$

Comparing the values of a_v from equation (4.12) to the above results one can get

$$\frac{b^v}{v!}e^{b^2} = \frac{1}{2^v v!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+2bx} \mathrm{H}_v(x)\, dx$$

Or
$$\int_{-\infty}^{\infty} e^{-x^2+2bx} \mathrm{H}_v(x)\, dx = 2^v b^v e^{b^2} \sqrt{\pi}$$

4.3. RECURRENCE RELATIONS OF HERMITE'S POLYNOMIALS

The recurrence relations of Hermite's polynomials to be established are

(*a*) $$\mathbf{2v\, H_{v-1}(x) = H'_v(x)}$$

Differentiating equation (4.2) with respect to x, one can obtain

$$2t\, e^{2xt-t^2} = \sum_{v=0}^{\infty} \frac{\mathrm{H}'_v(x)t^v}{v!}$$

Or
$$2\sum_{v=0}^{\infty} \frac{\mathrm{H}_v(x)t^{v+1}}{v!} = \sum_{v=0}^{\infty} \mathrm{H}'_v(x)t^v$$

Comparing the coefficients of $\frac{t^v}{v!}$ on both sides one will get

$$2v\, \mathrm{H}_{v-1}(x) = \mathrm{H}'_v(x) \qquad \text{...(4.14)}$$

(*b*) $$\mathbf{2x H_v(x) = 2v H_{v-1}(x) + H_{v+1}(x)}$$

Differentiating equation (4.2) with respect to t, one can obtain

$$2(x-t)e^{2xt-t^2} = \sum_{v=0}^{\infty} v \frac{\mathrm{H}_v(x)t^{v-1}}{v!}$$

$$2(x-t)\sum_{v=0}^{\infty} \frac{\mathrm{H}_v(x)t^v}{v!} = \sum_{v=0}^{\infty} \frac{\mathrm{H}_v(x)t^{v-1}}{(v-1)!}$$

$$2x\sum_{v=0}^{\infty} \frac{\mathrm{H}_v(x)t^v}{v!} - 2\sum_{v=0}^{\infty} \frac{\mathrm{H}_v(x)t^{v+1}}{v!} = \sum_{v=0}^{\infty} \frac{\mathrm{H}_v(x)t^{v-1}}{(v-1)\,!}$$

Comparing the coefficients of $\frac{t^v}{v!}$ on both sides one will get

$$2xH_v(x) - 2vH_{v-1}(x) = H_{v+1}(x)$$

Rearranging the terms, one will achieve the recurrence relation

$$2x\,H_v(x) = 2vH_{v-1}(x) + H_{v+1}(x) \quad \text{...(4.15)}$$

(*c*) $\mathbf{H'_v(x) = 2xH_v(x) - H_{v+1}(x)}$

Substituting the value of $2vH_{v-1}(x)$ from recurrence relation (4.14) in to the recurrence relation (4.15), one gets

$$\mathbf{2xH_v(x) = H'_v(x) + H_{v+1}(x)}$$

Rearranging the terms, one will achieve the recurrence relation

$$\mathbf{H'_v(x) = 2x\,H_v(x) - H_{v+1}(x)} \quad \text{...(4.16)}$$

Example 4.4. Show that $\int_{-\infty}^{\infty} x^m e^{-x^2} H_n(x)\,dx = 0$ **for any integer,** $0 \le m \le n-1$**.**

Solution : From equation (4.11) one knows that

$$f(x) = \sum_{v=0}^{\infty} a_v H_v(x)$$

Here $f(x) = x^m$, Hence the above eqn. can be written as

$$x^m = \sum_{v=0}^{\infty} a_v H_v(x)$$

Multiply the above eqn. with $e^{-x^2} H_n(x)$ and integrating between the limits $-\infty$ to ∞, one gets

$$\int_{-\infty}^{\infty} e^{-x^2} x^m H_n(x)\,dx = \sum_{v=0}^{\infty} a_v \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_v(x)\,dx$$

Using the orthogonality of Hermite's polynomial, the above equation can be reduced to

$$\int_{-\infty}^{\infty} e^{-x^2} x^m H_n(x)\,dx = \sum_{v=0}^{\infty} a_v 2^v v! \sqrt{\pi}\,\delta_{nv} = 0 \quad \begin{array}{l} \textit{if } n \ne v \\ \forall\ m \text{ lying between } 0 \le m \le n-1 \end{array}$$

for $n = v$ the eqn. becomes

$$\int_{-\infty}^{\infty} e^{-x^2} x^m H_n(x)\,dx = a_n 2^n n! \sqrt{\pi}$$

Thus

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^m H_n(x) dx$$

Example 4.5. **Prove that $H_0(x) = 1$**

Solution : Considering the generating function of Hermite's polynomial, from eqn. (4.2) and subtituting $v = 0$, one can find

$$e^{-2xt-t^2} = \frac{H_0(x) t^0}{0!}$$

$$e^{-2xt-t^2} = H_0(x) t^0$$

Expanding the LHS of above eqn., one gets

$$H_0(x) t^0 = 1 - 2xt - t^2 + \ldots\ldots$$

Comparing the powers of t^0 on both sides one gets

$$H_0(x) = 1$$

Example 4.6. **Prove that $H_1(x) = 2x$**

Solution : From the recurrence relation (4.14) substituting $v = 1$, one gets

$$2.1\, H_0(x) = H_1'(x)$$

or

$$\frac{d H_1(x)}{dx} = 2.1.1 = 2$$

Integrating both sides *w.r.t.* x, one gets

$$\int \frac{d H_1(x)}{dx} dx = 2\int dx$$

or

$$H_1(x) = 2x$$

Example 4.7. **Prove that $H_2(x) = 4x^2 - 2$.**

Solution : From the recurrence relation (4.16) substituting $v = 1$, one gets

$$H_1'(x) = 2xH_1(x) - H_2(x)$$

Substituting $H_1(x) = 2x$ and rearranging the terms, above equation could be rewritten as

$$H_2(x) = (2x) . (2x) - \frac{d}{dx}(2x)$$

or

$$H_2(x) = 4x^2 - 2$$

5. LAGUERRE DIFFERENTIAL EQUATION AND ITS SOLUTION

The differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ky = 0 \qquad \ldots(5.1)$$

is called Laguerre's differential equation of order k and its solutions are called Laguerre's functions. The differential equation appears in the radial part solution of one electron atoms, like hydrogen. The equation can be rewritten as

$$\frac{d^2y}{dx^2}+\frac{(1-x)}{x}\frac{dy}{dx}+\frac{k}{x}y = 0 \quad \text{...(5.2)}$$

Such that $P(x)=\frac{1-x}{x}$ and $Q(x)=\frac{k}{x}$ becomes infinite at $x = 0$ making it a singular point. The next step is to check the nature of singularity using the relations

$$\lim_{x\to 0}(x-0)\frac{1-x}{x} = 1 = \textit{Finite}$$

and

$$\lim_{x\to 0}(x-0)^2\frac{k}{x} = \lim_{x\to 0} kx = 0 = \textit{Finite}$$

Hence $x = 0$ is a regular singular point and thus the equation can be solved using Frobenius method. The solution to equation (5.1) is given as

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

So that
$$y' = \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1}$$

and
$$y'' = \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$$

Substituting the values of y, y', y'' in equation (5.1), one can obtain

$$x\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}+(1-x)\sum_{n=0}^{\infty} a_n (n+\alpha)\, x^{n+\alpha-1}+k\sum_{n=0}^{\infty} a_n x^{n+\alpha}=0$$

Or $$\sum_{n=0}^{\infty} a_n (n+\alpha)\,(n+\alpha-1) x^{n+\alpha-1}+\sum_{n=0}^{\infty} a_n (n+\alpha)\, x^{n+\alpha-1}-\sum_{n=0}^{\infty} a_n (n+\alpha)\, x^{n+\alpha}+k\sum_{n=0}^{\infty} a_n\, x^{n+\alpha} = 0$$

Or $$\sum_{n=0}^{\infty} a_n ((n+\alpha)(n+\alpha-1)+(n+\alpha))x^{n+\alpha-1} -\sum_{n=0}^{\infty} a_n\,(n+\alpha-k)\, x^{n+\alpha} = 0$$

Or $$\sum_{n=0}^{\infty} a_n (n+\alpha)^2\, x^{n+\alpha-1}-\sum_{n=0}^{\infty} a_n (n+\alpha-k)\, x^{n+\alpha}=0$$

Equating the coefficients of lowest power of x, *i.e.* $x^{\alpha-1}$ on both sides one can obtain

$$a_0\,\alpha^2 = 0$$

$a_0 \neq 0$, hence $\alpha^2 = 0$, which is an indicial equation, which provides the value of α as

$$\alpha = 0$$

Comparing the coefficients of $x^{\alpha+n}$, one can obtain

$$a_{n+1}\,(n + 1 + \alpha)^2 - a_n\,(n + \alpha - k) = 0$$

Or
$$a_{n+1} = \frac{(n+\alpha-k)}{(n+1+\alpha)^2}\,a_n \qquad \text{...(5.3)}$$

Substituting $\alpha = 0$, equation (5.3) becomes

$$a_{n+1} = \frac{(n-k)}{(n+1)^2}\,a_n$$

Substituting $n = 0, 1, 2, 3, 4$ ...., one can obtain the values of $a_n's$ in terms of a_0 as

$$a_1 = -ka_0$$

$$a_2 = \frac{(1-k)}{2^2}a_1 = (-1)^2\frac{k(k-1)}{2^2}a_0$$

$$a_3 = \frac{(2-k)}{3^2}a_1 = (-1)^3\frac{k(k-1)(k-2)}{3^2\,.\,2^2}a_0$$

$$a_{n+1} = (-1)^{n+1}\frac{k(k-1)(k-2)...(k-n)}{n^2...3^2.2^2}a_0 = (-1)^{n+1}\frac{k(k-1)(k-2)...(k-n)}{(n!)^2}a_0$$

Or
$$a_{n+1} = (-1)^{n+1}\frac{k(k-1)(k-2)...(k-n+1)(k-n)!}{(n!)^2(k-n)!}\,a_0$$

Or
$$a_{n+1} = (-1)^{n+1}\frac{k!}{(n!)^2(k-n)!}\,a_0$$

So that the generalized solution of Laguerre's equation could be written as

$$y = a_0\sum_{n=0}^{\infty}\frac{(-1)^{n+1}k!}{(n!)^2(k-n)!}x^n$$

By considering $a_0 = -\,k\,!$

The equation becomes
$$y = \sum_{n=0}^{\infty}\frac{(-1)^n(k!)^2}{(n!)^2(k-n)!}x^n \qquad \text{...(5.4)}$$

Equation (5.4) is the expression for Laguerre's functions and are represented by $L_k\,(x)$

5.1. GENERATING FUNCTIONS OF LAGUERRE FUNCTIONS

The generating function of Laguerre Functions is

$$\boldsymbol{g(x,t) = \frac{1}{1-t} e^{-\frac{xt}{1-t}} = \sum_{k=0}^{\infty} \frac{L_k(x)\, t^k}{k!}} \qquad ...(5.5)$$

To verify it consider the expansion of $\frac{1}{1-t} e^{-\frac{xt}{1-t}}$ as

$$\frac{1}{1-t} e^{-\frac{xt}{1-t}} = \frac{1}{1-t}\left(1 - \frac{xt}{1-t} + \frac{x^2t^2}{2!(1-t)^2} - \frac{x^3t^3}{3!(1-t)^3} + ... + \frac{(-1)^n x^n t^n}{n!(1-t)^n}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{n!\,(1-t)^{n+1}}$$

that can be expanded further by expanding the term $(1-t)^{-n-1}$ as

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{n!} (1-t)^{-n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{n!}$$

$$\left(1 + (n+1)\, t + \frac{(n+1)\,(n+2)}{2!} t^2 + ... + \frac{(n+1)\,(n+2)\, ...\, (n+l)}{l!} t^l + ...\right)$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^n x^n t^{n+l}}{n!} \frac{(n+1)\,(n+2)\, ...\, (n+l)\, n!}{n!\, l!}$$

Or
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{n!} (1-t)^{-n-1} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^n x^n t^{n+l}}{n!} \frac{(n+l)!}{n!\, l!} \qquad ...(5.6)$$

Substituting $n + l = k$, equation (5.6) becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{n!} (1-t)^{-n-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n x^n t^k}{n!} \frac{k!}{n!(k-n)!} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^n x^n t^k}{k!\,(n!)^2} \frac{(k!)^2}{(k-n)!} \qquad ...(5.7)$$

Comparing the results of equation (5.7) with equation (5.4), one can obtain

$$\frac{1}{1-t} e^{-\frac{xt}{1-t}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} L_k(x)$$

5.2. RODRIGUE'S FORMULA FOR LAGUERRE FUNCTIONS

The Rodrigue's formula or the differential representation for Laguerre functions is given as

$$L_k(x) = (-1)^k e^x \frac{d^k}{dx^k}(x^k e^{-x}) \qquad ...(5.8)$$

To prove it consider a function $g_k(x) = x^k e^{-x}$ and differentiate it k times with respect to x using Libnitz formula, such that

$$\frac{d^k(x^k e^{-x})}{dx^k} = \sum_{r=0}^{\infty} {}^k C_r \frac{d^{k-r} x^k}{dx^{k-r}} \frac{d^r e^{-x}}{dx^r} = \sum_{k=0}^{\infty} \frac{k!}{(k-r)!\,r!}\; k(k-1)(k-2)\ldots(k-r+1)\, x^{k-r} (-1)^r e^{-x}$$

Or $$\frac{d^k(x^k e^{-x})}{dx^k} = \sum_{r=0}^{\infty} \frac{k!}{(k-r)!\,r!} \frac{k(k-1)(k-2)\ldots(k-r+1)(k-r)!}{(k-r)!} x^{k-r} (-1)^r e^{-x}$$

Or $$\frac{d^k(x^k e^{-x})}{dx^k} = \sum_{r=0}^{\infty} \frac{(-1)^r (k!)^2}{((k-r)!)^2 r!} x^{k-r} e^{-x} \qquad \text{...(5.9)}$$

Substituting $k - r = n$, and multiplying equation (5.9) with e^x the equation becomes

$$e^x \frac{d^k(x^k e^{-x})}{dx^k} = (-1)^k \sum_{r=0}^{\infty} \frac{(-1)^n (k!)^2}{(n!)^2 (k-n)!} x^n$$

Or $$(-1)^k e^x \frac{d^k(x^k e^{-x})}{dx^k} = L_k(x)$$

5.3. ORTHOGONALITY OF LAGUERRE FUNCTIONS

The orthogonality of Laguerre Functions is given as

$$\int_0^{\infty} e^{-x} L_k(x) L_l(x)\, dx = (k!)^2 \delta_{kl}$$

And can be deduced by putting equation (5.1) in Sturm Liouville form as

$$\frac{d}{dx}\left(e^{-x} \frac{d\,L_k(x)}{dx}\right) + ke^{-x} L_k(x) = 0 \qquad \text{...(5.10)}$$

Similarly

$$\frac{d}{dx}\left(e^{-x} \frac{d\,L_l(x)}{dx}\right) + le^{-x} L_l(x) = 0 \qquad \text{...(5.11)}$$

Multiplying equation (5.10) with $L_l(x)$ and equation (5.11) with $L_k(x)$, subtracting and integrating between the limits 0 to ∞, one can get

$$\int_0^{\infty} \left[L_l(x) \frac{d}{dx}\left(e^{-x} \frac{dL_k(x)}{dx}\right) - L_k(x) \frac{d}{dx}\left(e^{-x} \frac{dL_l(x)}{dx}\right) \right] dx + (k-l) \int_0^{\infty} e^{-x} L_k(x)\, L_l(x)\, dx = 0$$

Or $$\int_0^{\infty} \frac{d}{dx} \left[L_l(x)\left(e^{-x} \frac{d\,L_x(x)}{dx}\right) - L_k(x)\left(e^{-x} \frac{d\,L_l(x)}{dx}\right) \right] dx + (k-l) \int_0^{\infty} e^{-x} L_k(x) L_l(x)\, dx = 0$$

Or $\left[L_l(x)\left(e^{-x}\frac{d\,L_k(x)}{dx}\right) - L_k(x)\left(e^{-x}\frac{dL_l(x)}{dx}\right)\right]_0^\infty + (k-l)\int_0^\infty e^{-x}L_k(x)L_l(x)\,dx = 0$

The first term in the square brackets becomes zero at both the limits, concluding

$$(k-l)\int_0^\infty e^{-x}L_k(x)\,L_l(x)\,dx = 0$$

Thus there are two conclusions from the above relation, one is

Either $(k-l) = 0$

Or $$\int_0^\infty e^{-x}L_k(x)\,L_l(x)\,dx = 0 \qquad ...(5.12)$$

that leads to the conclusion that, $\int_0^\infty e^{-x}L_k(x)L_l(x)\,dx = 0$, when $l \neq k$, When $l = m$, the relation can be evaluated from the generating function (5.5), after multiply with e^{-x} as follows :

$$\sum_{k=0}^\infty \frac{t^k}{k!}L_k(x) = \frac{1}{1-t}e^{-xt/1-t}$$

$$\left(\frac{t^k}{k!}\right)^2\int_0^\infty e^{-x}L_k^2(x)\,dx = \frac{1}{(1-t)^2}\int_0^\infty e^{-x-\frac{2xt}{1-t}}\,dx = \frac{1}{(1-t)^2}\int_0^\infty e^{-x\frac{1+t}{1-t}}dx$$

Or $$\frac{t^{2k}}{(k!)^2}\int_0^\infty e^{-x}L_k^2(x)\,dx = -\frac{1}{(1-t)^2}\left.\frac{e^{-x\frac{1+t}{1-t}}}{\left(\frac{1+t}{1-t}\right)}\right|_0^\infty = -\frac{1}{(1-t)^2}\left(\frac{1-t}{1+t}\right)(0-1)$$

$$\frac{t^{2k}}{(k!)^2}\int_0^\infty e^{-x}L_k^2(x)\,dx = \frac{1}{(1-t^2)} = \sum_{k=0}^\infty t^{2k}$$

Comparing the coefficients on both sides one can achieve the result

$$\int_0^\infty e^{-x}L_k^2(x)\,dx = (k!)^2 \qquad ...(5.13)$$

Equation (5.12) and (5.13) can be written in the combined form as

$$\int_0^\infty e^{-x}L_k(x)L_l(x)\,dx = (k!)^2\,\delta_{k!} \qquad ...(5.14)$$

5.4. RECURRENCE RELATIONS OF LAGUERRE FUNCTIONS

(*a*) $$k\,L_{k-1}(x) = kL'_{k-1}(x) - L'_k(x) \qquad ...(5.15)$$

Differentiating equation (5.5) with respect to *x*, one gets

$$-\frac{t}{(1-t)^2}e^{-\frac{xt}{1-t}} = \sum_{k=0}^{\infty}\frac{L'_k(x)t^k}{k!}$$

Or $$-\frac{t}{1-t}\sum_{k=0}^{\infty}\frac{L_k(x)t^k}{k!} = \sum_{k=0}^{\infty}\frac{L'_k(x)t^k}{k!}$$

Or $$-\sum_{k=0}^{\infty}\frac{L_k(x)t^{k+1}}{k!} = \sum_{k=0}^{\infty}\frac{L'_k(x)t^k}{k!} - \sum_{k=0}^{\infty}\frac{L'_k(x)t^{k+1}}{k!}$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides, one gets

$$-kL_{k-1}(x) = L'_k(x) - kL'_{k-1}(x)$$

Or $$kL_{k-1}(x) = kL'_{k-1}(x) - L'_k(x)$$

(*b*) $$\mathbf{L_{k+1}(x) + (x - 2k - 1)\,L_k(x) + k^2\,L_{k-1}(x) = 0} \qquad ...(5.16)$$

Differentiating equation (5.5) with respect to *t*, one gets

$$\left(-\frac{x}{(1-t)^2} - \frac{xt}{(1-t)^3} + \frac{1}{(1-t)^2}\right)e^{-\frac{xt}{1-t}} = \sum_{k=0}^{\infty}\frac{L_k(x)t^{k-1}}{(k-1)!}$$

Or $$\left(\frac{-x+1-t}{(1-t)^3}\right)e^{-\frac{xt}{1-t}} = \sum_{k=0}^{\infty}\frac{L_k(x)\,t^{k-1}}{(k-1)!}$$

Or $$\left(\frac{-x+1-t}{(1-t)^2}\right)\sum_{k=0}^{\infty}\frac{L_k(x)t^k}{k!} = \sum_{k=0}^{\infty}\frac{L_k(x)t^{k-1}}{(k-1)!}$$

Or $$(-x+1-t)\sum_{k=0}^{\infty}\frac{L_k(x)t^k}{k!} = (1-t)^2\sum_{k=0}^{\infty}\frac{L_k(x)t^{k-1}}{(k-1)!}$$

Or $$(-x+1-t)\sum_{k=0}^{\infty}\frac{L_k(x)t^k}{k!} = (1-2t+t^2)\sum_{k=0}^{\infty}\frac{L_k(x)t^{k-1}}{(k-1)!}$$

Or $$(-x+1)\sum_{k=0}^{\infty}\frac{L_k(x)t^k}{k!} - \sum_{k=0}^{\infty}\frac{L_k(x)\,t^{k+1}}{k!} = \sum_{k=0}^{\infty}\frac{L_k(x)t^{k-1}}{(k-1)!} - 2\sum_{k=0}^{\infty}\frac{L_k(x)\,t^k}{(k-1)!} + \sum_{k=0}^{\infty}\frac{L_k(x)\,t^{k+1}}{(k-1)!}$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides, one gets

$$(1-x)\,L_k(x) + kL_{k-1}(x) = L_{k+1}(x) - 2kL_k(x) + k(k+1)\,L_{k-1}(x)$$

Rearranging the terms one gets

$L_{k+1}(x) + (x - 2k - 1)\,L_k(x) + k^2\,L_{k-1}(x) = 0$

6. BESSEL'S DIFFERENTIAL EQUATION AND IT'S SOLUTION

The differential equation

$$\boldsymbol{x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)\,y = 0} \qquad \text{...(6.1)}$$

is called Bessel's differential equation of order m and its solutions are called Bessel's functions. The equation has many applications in solving the physical problems and hence is of great importance. It finds its applications in solution of Laplace equation and Helmholtz equation of a quantum free particle. The equation can be rewritten as

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{(x^2 - m^2)}{x^2}\,y = 0 \qquad \text{...(6.2)}$$

Such that $P(x) = \frac{1}{x}$ and $Q(x) = \frac{(x^2 - m^2)}{x^2}$ becomes infinite at $x = 0$ making it a singular point. The next step is to check the nature of singularity using the relations

$$\lim_{x \to x_0} (x - x_0)\,P(x)$$

and
$$\lim_{x \to 0} (x - x_0)^2\,Q(x)$$

for Bessel's equation. Thus

$$\lim_{x \to 0} (x - 0)\frac{1}{x} = 1 = \textit{Finite}$$

and
$$\lim_{x \to 0} (x - 0)^2 \frac{(x^2 - m^2)}{x^2} = \lim_{x \to 0} (x^2 - m^2) = -m^2 = \textit{Finite}$$

Hence $x = 0$ is a regular singular point and thus the equation can be solved using Frobenius method. The solution to equation (6.1) is given as

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

So that
$$y' = \sum_{n=0}^{\infty} a_n (n+\alpha)\,x^{n+\alpha-1}$$

and $$y'' = \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)\, x^{n+\alpha-2}$$

Substituting the values of y, y', y'' in equation (6.1), one can obtain

$$x^2 \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)\, x^{n+\alpha-2} + \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1} + (x^2 - m^2) \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0$$

Or $$\sum_{n=0}^{\infty} a_n\, (n+\alpha)(n+\alpha-1)\, x^{n+\alpha} + \sum_{n=0}^{\infty} a_n (n+\alpha)\, x^{n+\alpha} + \sum_{n=0}^{\infty} a_n\, x^{n+\alpha+2} - m^2 \sum_{n=0}^{\infty} a_n\, x^{n+\alpha} = 0$$

Or $$\sum_{n=0}^{\infty} a_n ((n+\alpha)\,(n+\alpha-1) + (n+\alpha) - m^2)\, x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or $$\sum_{n=0}^{\infty} a_n ((n+\alpha)\,(n+\alpha-1) + (n+\alpha) - m^2)\, x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Or $$\sum_{n=0}^{\infty} a_n ((n+\alpha)^2 - m^2)\, x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = 0$$

Equating the coefficients of lowest power of x, *i.e.* x^a on both sides one can obtain

$$a_0\,(\alpha^2 - m^2) = 0$$

As $a_0 \neq 0$, hence $\alpha^2 - m^2 = 0$, which is an indicial equation, which provides the value of α as

$$\alpha = \pm m$$

Comparing the coefficients of $x^{\alpha+1}$, one can obtain

$$a_1\,((\alpha + 1)^2 - m^2) = 0$$

As $\alpha = \pm m$, hence a_1 should be equal to zero

Equating the coefficients of $x^{n+\alpha+2}$, it could be retrieved as

$$a_{n+2}\,((n + 2 + \alpha)^2 - m^2) + a_n = 0$$

Or $$a_{n+2} = -\frac{a_n}{(n+2+\alpha)^2 - m^2}$$

Substituting $n = 0, 1, 2, 3, 4$ ..., one can obtain the values of $a'_n s$ in terms of a_0 and a_1 as

$$a_2 = -\frac{1}{(2+\alpha)^2 - m^2} a_0, \qquad a_3 = -\frac{1}{(3+\alpha)^2 - m^2} a_1 = 0,$$

$$a_4 = -\frac{1}{(4+\alpha)^2 - m^2}\, a_2 = \frac{(-1)^2 a_0}{((4+\alpha)^2 - m^2)((2+\alpha)^2 - m^2)} \qquad a_5 = \frac{1}{(5+\alpha)^2 - m^2} a_3 = 0\,,$$

$$a_6 = \frac{1}{(6+\alpha)^2 - m^2} a_4 = \frac{(-1)^3 a_0}{((6+\alpha)^2 - m^2)((4+\alpha)^2 - m^2)((2+\alpha)^2 - m^2)} \quad a_7 = \frac{1}{(7+\alpha)^2 - m^2} a_5 = 0$$

And the generalized term could be written as

$$a_{2n} = \frac{(-1)^n a_0}{((2n+\alpha)^2 - m^2)..((6+\alpha)^2 - m^2)((4+\alpha)^2 - m^2)((2+\alpha)^2 - m^2)}$$

Substituting the value of $\alpha = m$, the term could be rewritten as

$$a_{2n} = \frac{(-1)^n a_0}{((2n+m)^2 - m^2)..((6+m)^2 - m^2)((4+m)^2 - m^2)((2+m)^2 - m^2)}$$

Or
$$a_{2n} = \frac{(-1)^n a_0}{2n...6.4.2.\ (2+2m)\ (4+2m)\ (6+2m)\ ....\ (2n+2m)}$$

Or
$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}.n....3.2.1.\ (1+m)\ (2+m)\ (3+m)\ ....\ (n+m)}$$

Multiplying and dividing the expression of a_{2n} obtained above with m!, the term could be rewritten as

$$a_{2n} = \frac{(-1)^n m!}{2^{2n} n!\,(n+m)!} a_0$$

Substituting the value of $\alpha = -m$, the second coefficient could be obtained as

$$a_{2n} = \frac{(-1)^n (-m)!}{2^{2n} n!(n-m)!} a_0'$$

So that the generalized solution of Bessel's equation could be written as

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n m!}{2^{2n} n!(n+m)!} x^{2n+m} a_0' \sum_{n=0}^{\infty} \frac{(-1)^n (-m)!}{2^{2n} n!(n-m)!} x^{2n-m}$$

Or
$$\boldsymbol{y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 2^m m!}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m} + a_0' \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-m} (-m)!}{n!(n-m)!} \left(\frac{x}{2}\right)^{2n-m}}$$

By considering $a_0 = \frac{1}{2^m m!}$ and $a_0' = \frac{1}{2^{-m}(-m)!}$, the expression for y can be expressed as follows :

$$\boldsymbol{y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{2n+m} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n-m)!} \left(\frac{x}{2}\right)^{2n-m}} \quad ...(6.3)$$

But if $|\,m\,| > 1$, $(-m + 1)$ is negative, the factorial for negative numbers are not defined. This dilemma however can be circumvented by working with the definition of gamma function for non integers, nonetheless the difficulty still pursues, if m is a negative integer.

6.1. BESSEL'S FUNCTION

The two solutions given in equation (4.3) of the Bessel's differential equation are known as Bessel's Function. Thus the first term

$$y_1 = J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!}\left(\frac{x}{2}\right)^{2n+m} \qquad ...(6.4)$$

is called the Bessel's function for positive m values and the second term of the solution of Bessel's Equation

$$y_2 = J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n-m)!}\left(\frac{x}{2}\right)^{2n-m} \qquad ...(6.5)$$

are the Bessel's function for negative m values.

6.2. LINEAR DEPENDENCY OF THE TWO SOLUTIONS OF BESSEL'S EQUATION

When $m = 0$, $J_m(x) = J_0(x)$ and also $J_{-m}(x) = J_0(x)$, the two solutions y_1 and y_2 became same, and hence are not linearly dependent. If m is an integer again they are not linearly independent. To prove it consider

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-m)!}\left(\frac{x}{2}\right)^{2n-m}$$

Let $n' = n - m$, so that the above equation becomes

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n'+m}}{(n'+m)\,!\,n'!}\left(\frac{x}{2}\right)^{2n'+m}$$

$$J_{-m}(x) = (-1)^m J_m(x) \qquad ...(6.6)$$

Thus the two solutions of Bessel's differential equation are linearly dependent. If m is not an integer $J_{-m}(x) \neq J_m(x)$ and hence are linearly independent. The proof of the same will be discussed after defining Wronskian.

6.3. WRONSKIAN AND THE SECOND SOLUTION

Frobenius method yields only one solution. It fails to give a second, linearly independent solution. In that case, the second solution can be obtained by some other method. The other method of getting second solution is using the Wronskian. Wronskian of two linearly independent solutions of a differential equation is given as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Here primes denotes differentiation with respect to x. The first derivative of wronskian with respect to x is then given as

$$\frac{dW(x)}{dx} = y_1\, y''_2 - y_2\, y''_1 \qquad ...(6.7)$$

Since y_1 and y_2 are the solutions to equation (2.5), thus

$$y''_1 = -P(x)\,y'_1 - Q(x)\,y_1 \qquad ...(6.8)$$

And

$$y''_2 = -P(x)\,y'_2 - Q(x)\,y_2 \qquad ...(6.9)$$

Substituting equation (6.8) and (6.9) in eqution (6.7) one gets

$$\frac{d\,W(x)}{dx} = y_1(-P(x)\,y'_2 - Q(x)\,y_2) - y_2(-P(x)\,y'_1 - Q(x)\,y_1)$$

Or
$$\frac{d\,W(x)}{dx} = -P(x)\,(y_1 y'_2 - y_2 y'_1)$$

Or
$$\frac{d\,W(x)}{dx} = -P(x)\,W(x)$$

Using the method of variable separation and integrating the two sides, one gets

$$\ln W(x) = -\int P(x)\,dx$$

Or
$$W(x) = W(a)\,e^{-\int P(x)\,dx} \qquad ...(6.10)$$

Equation (6.10) can now be used to get the second solution y_2 if y_1 is known by the following method.

$$W(x) = y_1 y'_2 - y_2 y'_1 = y_1^2\left(\frac{y'_2}{y_1} - \frac{y'_1}{y_1^2}\right) = y_1^2\,\frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

Thus equation (6.10) can be rewritten as

$$y_1^2\,\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a)\,e^{-\int P(x)\,dx}$$

Or
$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a)\frac{e^{-\int P(x)dx}}{y_1^2} \qquad ...(6.11)$$

Integrating equation (6.11) *w.r.t.* x one gets

$$\frac{y_2}{y_1} = \int W(a)\,\frac{e^{-\int P(x)dx}}{y_1^2}\,dx$$

Or
$$y_2 = y_1\int W(a)\frac{e^{-\int P(x)dx}}{y_1^2}\,dx \qquad ...(6.12)$$

That is the second solution to a general differential equation, whose second solution could not be obtained using Frobenius method.

Example 6.1. Find the second solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$$

And the first solution is given as $J_0(x)$

Solution : The given equation can be rewritten as

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

Thus $P(x) = \frac{1}{x}$, so that $e^{-\int P(x)dx} = e^{-\int dx/x} = e^{-\ln x} = \frac{1}{x}$, and y_2 is given as

$$y_2 = J_0(x)\int \frac{1}{x}\frac{1}{(J_0(x))^2}dx \qquad ...(6.13)$$

Here $$J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} = \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right)$$

So that equation (6.13) can be rewritten as

$$y_2 = \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right)\int \frac{1}{x}\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right)^{-2} dx$$

Or $$y_2 = \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right)\int \frac{1}{x}\left(1 + \frac{x^2}{2} + \frac{5x^4}{32} + ...\right)dx$$

Or $$y_2 = \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right)\int \left(\frac{1}{x} + \frac{x}{2} + \frac{5x^3}{32} + ...\right)dx$$

Or $$y_2 = \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right)\left(\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + ...\right)$$

$$y_2 = \ln x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right) + \frac{x^2}{4}\left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ...\right) + ...$$

6.4. J_m (x) AND J_{-m} (x) ARE LINEARLY INDEPENDENT FOR *m*, NOT AN INTEGER

If the wronskian of $J_m(x)$ and $J_{-m}(x)$ is non zero for m, a non integer, the Bessel's functions will be proved to be linearly independent, hence consider

$$\frac{d\,W(x)}{dx} = \frac{d}{dx}\left(J_m(x)\,J'_{-m}(x) - J'_m(x)\,J_{-m}(x)\right) = J_m(x)\,J''_{-m}(x) - J''_m(x)\,J_{-m}(x) \quad \text{...(6.14)}$$

Since $J_m(x)$ and $J_{-m}(x)$, both are the solutions of equation (6.1), thus

$$\frac{d^2 J_m(x)}{dx^2} = -\frac{1}{x}\frac{dJ_m(x)}{dx} - \left(1 - \frac{m^2}{x^2}\right) J_m(x) \quad \text{...(6.15)}$$

And

$$\frac{d^2 J_{-m}(x)}{dx^2} = -\frac{1}{x}\frac{dJ_{-m}(x)}{dx} - \left(1 - \frac{m^2}{x^2}\right) J_{-m}(x) \quad \text{...(6.16)}$$

Substituting equations (6.15) and (6.16) in equation (6.14), one gets

$$\frac{d\,W(x)}{dx} = J_m(x)\left(-\frac{1}{x}\frac{d\,J_{-m}(x)}{dx} - \left(1 - \frac{m^2}{x^2}\right) J_{-m}(x)\right) - J_{-m}(x)$$

$$\left(-\frac{1}{x}\frac{d\,J_m(x)}{dx} - \left(1 - \frac{m^2}{x^2}\right) J_m(x)\right)$$

Or

$$\frac{d\,W(x)}{dx} = \left(-\frac{J_m(x)}{x}\frac{d\,J_{-m}(x)}{dx} - \frac{J_{-m}(x)}{x}\frac{d\,J_m(x)}{dx}\right)$$

Or

$$\frac{d\,W(x)}{dx} = -\frac{W(x)}{x}$$

Or

$$x\frac{d\,W(x)}{dx} + W(x) = 0$$

Or

$$\frac{d(xW(x))}{dx} = 0$$

Or

$$xW(x) = constant$$

Or

$$W(x) = \frac{constant}{x} \quad \text{...(6.17)}$$

Thus $W(x) \neq 0$, proving that for m, a non integer, the two solutions to Bessel's equation are linearly independent.

To evaluate the constant consider the wronskian when $n \to 0$ in the expression of $J_m(x)$ and $J_{-m}(x)$ such that

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m$$

$$J_{-m}(x) = \frac{1}{(-m)!}\left(\frac{x}{2}\right)^{-m}$$

$$J'_{m}(x) = \frac{2}{(m-1)!}\left(\frac{x}{2}\right)^{m-1}$$

$$J'_{-m}(x) = \frac{1}{2(-m+1)\,!}\left(\frac{x}{2}\right)^{-m-1}$$

The Wronskian becomes

$$W(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^{m}\frac{1}{2(-m+1)\,!}\left(\frac{x}{2}\right)^{-m-1} - \frac{1}{(-m)!}\left(\frac{x}{2}\right)^{-m}\frac{2}{(m-1)!}\left(\frac{x}{2}\right)^{m-1}$$

Or $$W(x) = \frac{1}{x}\left(\frac{1}{m!(-m+1)!} - \frac{1}{(-m)!(m-1)!}\right)$$

Or $$W(x) = \frac{1}{x}\left(\frac{1}{\Gamma(m+1)\Gamma(-m)} - \frac{1}{\Gamma(m)\Gamma(-m+1)}\right)$$

Using the property of $\Gamma(m+1) = m\Gamma(m)$, the above relation could be rewritten as

$$W(x) = \frac{1}{x}\left(\frac{1}{m\Gamma(m)\Gamma(-m)} + \frac{1}{m\Gamma(m)\,\Gamma(-m)}\right) = \frac{2}{x}\frac{1}{m\Gamma(m)\,\Gamma(-m)}$$

Using the property of gamma function for non integers

$$\Gamma(m)\,\Gamma(-m) = -\frac{\pi}{m\sin\pi m}$$

The expression for W (x) can be rewritten as

$$W(x) = -\frac{2}{x}\frac{m\sin\pi m}{m\pi} = -\frac{2}{x}\frac{\sin\pi m}{\pi} \qquad \text{...(6.18)}$$

Comparing equation (6.17) and (6.18), one gets

$$\text{Constant} = -\frac{2\sin\pi m}{\pi}$$

6.5. GENERATING FUNCTION OF BESSEL'S FUNCTION

The generating function of Bessel's equation is given as

$$\boldsymbol{g(x, t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}} \qquad \text{...(6.19)}$$

To prove it separate the exponential of the generating function into two parts

$$g(x, t) = e^{\frac{x}{2}t}, e^{-\frac{x}{2t}}$$

Expand the exponential terms using Taylor series expansion

$$g(x, t) = \sum_{n=-\infty}^{\infty} \frac{\left(\frac{x}{2}t\right)^n}{n!} \sum_{n'=0}^{\infty} (-1)^{n'} \frac{\left(\frac{x}{2t}\right)^{n'}}{n'!}$$

Or $$g(x, t) = \sum_{n=-\infty}^{\infty} \sum_{n'=0}^{\infty} (-1)^{n'} \left(\frac{x}{2}\right)^{n+n'} \frac{t^{n-n'}}{n!n'!} \quad ...(6.20)$$

Considering $n = n' + m$, equation (6.20) becomes

$$g(x,t) = \sum_{n'=0}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{n'} \left(\frac{x}{2}\right)^{2n'+m} \frac{t^m}{(n'+m)!n'!}$$

Or $$g(x, t) = \sum_{m=0}^{\infty} J_m(x) t^m$$

Comparing the results of equation (6.19) and (6.20) one gets

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} J_m(x) t^m \quad ...(6.21)$$

6.6. RECURRENCE RELATIONS OF BESSEL'S FUNCTION

The Bessel's function satisfy following recurrence relations

(*a*) $$\mathbf{J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)} \quad ...(6.22)$$

Consider the differentiation of generating function (6.21) of Bessel's Functions with respect to *t*

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} mJ_m(x) t^{m-1}$$

Or $$\frac{x}{2}\left(1+\frac{1}{t^2}\right) \sum_{m=-\infty}^{\infty} J_m(x) t^m = \sum_{m=-\infty}^{\infty} mJ_m(x) t^{m-1}$$

Or $$\frac{x}{2} \sum_{m=-\infty}^{\infty} J_m(x) t^m + \frac{x}{2} \sum_{m=-\infty}^{\infty} J_m(x)\, t^{m-2} = \sum_{m=-\infty}^{\infty} mJ_m(x)\, t^{m-1}$$

Comparing the coefficients t^{m-1} on both sides one gets

$$\frac{x}{2} J_{m-1}(x) + \frac{x}{2} J_{m+1}(x) = mJ_m(x)$$

Or $$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

(*b*) $$\mathbf{J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x)} \qquad \text{...(6.23)}$$

Consider the differentiation of generating function (6.21) of Bessel's Functions with respect to *x*

$$\frac{1}{2}\left(t - \frac{1}{t}\right) e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} J'_m(x) t^m$$

Or $$\frac{1}{2}\left(t - \frac{1}{t}\right) \sum_{m=-\infty}^{\infty} J_m(x) t^m = \sum_{m=-\infty}^{\infty} J'_m(x) t^m$$

Or $$\frac{1}{2} \sum_{m=-\infty}^{\infty} J_m(x) t^{m+1} - \frac{1}{2} \sum_{m=-\infty}^{\infty} J_m(x) t^{m-1} = \sum_{m=-\infty}^{\infty} J'_m(x) t^m$$

Comparing the coefficients t^m on both sides one gets

$$\frac{1}{2} J_{m-1}(x) - \frac{1}{2} J_{m+1}(x) = J'_m(x)$$

Or $$J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x)$$

(*c*) $$\mathbf{J_{m+1}(x) = \frac{m}{x} J_m(x) - J'_m(x)} \qquad \text{...(6.24)}$$

Subtracting equation (6.22) from equation (6.23), one gets

$$2J_{m+1}(x) = \frac{2m}{x} J_m(x) - 2J'_m(x)$$

Or $$J_{m+1}(x) = \frac{m}{x} J_m(x) - J'_m(x)$$

<u>Example 6.2.</u> Express $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$

Solution : Consider the recurrence relation

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

And let $m = 5$, such that

$$J_4(x) + J_6(x) = \frac{10}{x} J_5(x)$$

Or $$J_6(x) = \frac{10}{x} J_5(x) - J_4(x) \qquad \text{...(6.25)}$$

Again substituting $m = 4$ in equation (6.22), one gets

$$J_5(x) = \frac{8}{x} J_4(x) - J_3(x)$$

So that equation (6.25) becomes

$$J_6(x) = \frac{10}{x}\left(\frac{8}{x} J_4(x) - J_3(x)\right) - J_4(x)$$

Or
$$J_6(x) = \left(\frac{80}{x} - 1\right) J_4(x) - \frac{10}{x} J_3(x)$$

Substituting $m = 3$ in equation (6.22) and back substitution to $J_4(x)$ in above equation, one gets

$$J_6(x) = \left(\frac{80}{x} - 1\right)\left(\frac{6}{x} J_3(x) - J_2(x)\right) - \frac{10}{x} J_3(x)$$

Or
$$J_6(x) = \left(\frac{480}{x^2} - \frac{16}{x}\right) J_3(x) - \left(\frac{80}{x} - 1\right) J_2(x)$$

Proceeding in a same way one gets

$$J_6(x) = \left(\frac{3840}{x^5} - \frac{768}{x^3} + \frac{18}{x}\right) J_1(x) - \left(\frac{1920}{x^4} - \frac{144}{x^2} + 1\right) J_0(x)$$

6.7. DIFFERENTIAL FORM OF BESSEL'S FUNCTION

The differential form of Bessel's function is given as

$$J_m(x) = (-1)^m x^m . \left(\frac{1}{x}\frac{d}{dx}\right)^m J_0(x) \qquad ...(6.26)$$

This can be proved by using induction method, for that

Consider equation (6.23) and let $m = 0$, one gets

$$J_{-1}(x) - J_1(x) = 2J'_0(x)$$

Using (6.6) from $m = 1$, one gets

$$J_{-1}(x) = -J_1(x)$$

Hence the above equation becomes

$$-2 J_1(x) = 2J'_0(x)$$

Or
$$-J_1(x) = J'_0(x)$$

Thus equation (6.26) is true for $m = 0$, consider that the equation is also true for $m = n$, so that equation (6.26) can be rewritten as

$$J_n(x) = (-1)^n x^n . \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x) \qquad ...(6.27)$$

To prove that it is also true for $m = n + 1$, consider

$$(-1)^{n+1}x^{n+1}.\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} J_0(x) = (-1)^{n+1}x^{n+1}\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)^{n} J_0(x) \quad ...(6.28)$$

From equation (6.27) one gets

$$(-1)^n x^{-n} J_n(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^{n} J_0(x)$$

Hence equation (6.28) becomes

$$(-1)^{n+1}x^{n+1}.\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} J_0(x) = (-1)^{n+1}x^{n+1}\frac{1}{x}\frac{d}{dx}((-1)^n x^{-n} J_n(x))$$

Or $$(-1)^{n+1}x^{n+1}.\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} J_0(x) = -\frac{dJ_n(x)}{dx} + \frac{m}{x}J_n(x)$$

Using equation (6.24) one gets

$$(-1)^{n+1}x^{n+1}.\left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} J_0(x) = J_{n+1}(x)$$

Hence the relation is proved using the induction method.

6.8. ORTHOGONALITY OF BESSEL'S FUNCTION

The orthogonality relation for Bessel's function is given as

$$\int_0^1 x J_m(\alpha x) J_m(\beta x)\, dx = \frac{(J_{m+1}(\alpha x))^2}{2}\delta_{\alpha\beta}$$

Here $J_n(\alpha) = J_n(\beta) = 0$ as α and β are the roots of Bessel's Function.
To prove it consider Bessel's equation

$$x^2\frac{d^2 J_m(\alpha x)}{dx^2} + x\frac{dJ_m(\alpha x)}{dx} + (\alpha^2 x^2 - m^2)\, J_m(\alpha x) = 0 \quad ...(6.29)$$

And $$x^2\frac{d^2 J_m(\beta x)}{dx^2} + x\frac{dJ_m(\beta x)}{dx} + (\beta^2 x^2 - m^2)\, J_m(\beta x) = 0 \quad ...(6.30)$$

Multiply equation (6.29) with $\frac{J_m(\beta x)}{x}$ and (6.30) with $\frac{J_m(\alpha x)}{x}$ and subtract

$$\frac{J_m(\beta x)}{x}\left(x^2\frac{d^2 J_m(\alpha x)}{dx^2} + x\frac{dJ_m(\alpha x)}{dx}\right) - \frac{J_m(\alpha x)}{x}\left(x^2\frac{d^2 J_m(\beta x)}{dx^2} + x\frac{dJ_m(\beta x)}{dx}\right) + (\alpha^2 - \beta^2)\, x\, J_m(\alpha x)\, J_m(\beta x) = 0$$

Or $$\frac{d}{dx}\left(J_m(\beta x)x\frac{dJ_m(\alpha x)}{dx} - J_m(\alpha x)\, x\frac{dJ_m(\beta x)}{dx}\right) + (\alpha^2 - \beta^2)\, x\, J_m(\alpha x) J_m(\beta x) = 0$$

Integrating with respect to x between the limits 0 to 1 one gets

$$\int_0^1 \frac{d}{dx}\left(J_m(\beta x) x \frac{dJ_m(\alpha x)}{dx} - J_m(\alpha x) x \frac{dJ_m(\beta x)}{dx} \right) dx + (\alpha^2 - \beta^2) \int_0^1 x J_m(\alpha x) J_m(\beta x)\, dx = 0$$

Or $$\alpha J_m(\beta x) x \frac{dJ_m(\alpha x)}{dx} - \beta J_m(\alpha x)_x \frac{dJ_m(\beta x)}{dx}\Bigg|_0^1 + (\alpha^2 - \beta^2) \int_0^1 x J_m(\alpha x) J_m(\beta x)\, dx = 0$$

Or $$-\frac{(\alpha J_m(\beta) J'_m(\alpha) - \beta J_m(\alpha) J'_m(\beta))}{(\alpha^2 - \beta^2)} = \int_0^1 x J_m(\alpha x) J_m(\beta x)\, dx = 0 \qquad \text{...(6.31)}$$

The first term becomes 0 because $J_n(\alpha) = J_n(\beta) = 0$, therefore the above equation can be rewritten as

$$(\alpha^2 - \beta^2) \int_0^1 x J_m(\alpha x) J_m(\beta x)\, dx = 0$$

Thus there are two conclusions from the above relation, one is either $(\alpha^2 - \beta^2)$ is zero or $\int_0^1 x J_m(\alpha x)\, J_m(\beta x)\, dx$ is zero, that leads to the conclusion that, $\int_0^1 x J_m(\alpha x)\, J_m(\beta x)\, dx = 0$, when $\alpha \neq \beta$. When $\alpha = \beta$, the left hand side of equation (6.31) takes 0/0 form, thus L'hospital's rule can be applied to the LHS of equation (6.31) to get

$$-\lim_{\alpha \to \beta} \frac{(J_m(\beta) J'_m(\alpha) - (\beta J'_m(\alpha) J'_m(\beta))}{2\alpha} = -\lim_{\alpha \to \beta} \frac{-\beta J'_m(\alpha) J'_m(\beta)}{2\alpha}$$

As $J_m(\beta) = 0$

$$-\lim_{\alpha \to \beta} \frac{-\beta J'_m(\alpha) J'_m(\beta)}{2\alpha} = \frac{\beta J'_m(\beta) J'_m(\beta)}{2\beta} = \frac{1}{2}(J'_m(\beta))^2 \qquad \text{...(6.32)}$$

Using the recurrence relation (6.23) and considering $x = \beta$, and using $J_m(\beta) = 0$, one gets

$$J_{m+1}(\beta) = -J'_m(\beta)$$

So that equation (6.31) becomes

$$\int_0^1 x\, J_m(\alpha x)\, J_m(\beta x)\, dx = \frac{1}{2}(J'_m(\beta))^2 = \frac{1}{2}(J_{m+1}(\beta))^2$$

6.9. INTEGRAL REPRESENTATION OF BESSEL'S FUNCTION

Substituting $t = e^{i\theta}$ in equation (6.21), one gets

$$\sum_{m=-\infty}^{\infty} J_m(x)\, e^{im\theta} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}$$

Or $$e^{ix\,(\sin\theta)} = \sum_{m=-\infty}^{\infty} J_m(x)\,(\cos m\theta - i\sin m\theta)$$

Or $$\cos[x(\sin\theta)] + i\sin[x(\sin\theta)] = \sum_{m=-\infty}^{\infty} J_m(x)\,(\cos m\theta - i\sin m\theta)$$

Comparing the real and imaginary part on both sides, one gets

$$\cos[x(\sin\theta)] = \sum_{m=-\infty}^{\infty} J_m(x)\cos m\theta \qquad \sin[x(\sin\theta)] = \sum_{m=-\infty}^{\infty} J_m(x)\sin m\theta$$

$$\cos[x(\sin\theta)] = J_0(x) + \sum_{m=1}^{\infty} (J_m(x) + J_{-m}(x))\cos m\theta \qquad \sin[x(\sin\theta)] = \sum_{m=1}^{\infty} (J_m(x) - J_{-m}(x))\sin m\theta$$

$$\cos[x(\sin\theta)] = J_0(x) + \sum_{m=1}^{\infty} (J_m(x) + (-1)^m J_m(x))\cos m\theta \quad \sin[x(\sin\theta)] = \sum_{m=1}^{\infty} (J_m(x) - (-1)^m J_m(x))\sin m\theta$$

The terms on the left hand side, *i.e.* belonging to cos [x (sin θ)] will vanish for odd m and that of right side, *i.e.* belonging to sin [x (sin θ)] will vanish for even m, thus the terms could be rewritten as

$$\cos[x(\sin\theta)] = J_0(x) + 2\sum_{m=2\,(even)}^{\infty} J_m(x)\cos m\theta \qquad \text{...(6.33)}$$

And $$\sin[x(\sin\theta)] = 2\sum_{m=1\,(odd)}^{\infty} J_m(x)\sin m\theta \qquad \text{...(6.34)}$$

Multiplying equation (6.33) with cos nθ and integrating between the limits 0 to π, one will be able to get

$$\int_0^{\pi} \cos[x(\sin\theta)]\cos n\theta\, d\theta = J_0(x)\int_0^{\pi}\cos n\theta\, d\theta + 2\sum_{m=2\,(even)}^{\infty} J_m(x)\int_0^{\pi}\cos m\theta\cos n\theta\, d\theta$$

Using the integral formula $\int_0^{\pi}\cos m\theta\cos n\theta\, d\theta = \frac{\pi}{2}\delta_{mn}$, the above equation can be rewritten as

$$\int_0^{\pi}\cos[x(\sin\theta)]\cos m\theta\, d\theta = J_0(x)(0) + \pi\sum_{m=2(even)}^{\infty} J_m(x)\,\delta_{mn}$$

Or $$\int_0^{\pi}\cos[x(\sin\theta)]\cos n\theta\, d\theta = \pi J_n(x) \text{ if } n \text{ is even} \qquad \text{...(6.35)}$$

Similary, multiplying equation (6.34) with sin nθ and integrating between the limits 0 to π, one will be able to get

$$\int_0^{\pi}\sin[x(\sin\theta)]\sin n\theta\, d\theta = 2\sum_{m=1(even)}^{\infty} J_m(x)\int_0^{\pi}\sin m\theta\sin n\theta\, d\theta$$

Using the integral formula $\int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = \frac{\pi}{2} \delta_{mn}$, the above equation can be rewritten as

$$\int_0^{\pi} \sin[x(\sin\theta)] \sin m\theta \, d\theta = \pi \sum_{m=1\,(even)}^{\infty} J_m(x)\, \delta_{mn}$$

Or $$\int_0^{\pi} \sin[x(\sin\theta)] \sin m\theta \, d\theta = \pi J_n(x) \text{ if } n \text{ is odd} \qquad ...(6.36)$$

Adding equation (6.35) and (6.36) one gets

$$\pi J_n(x) = \int_0^{\pi} \cos[x(\sin\theta)] \cos n\theta \, d\theta + \int_0^{\pi} \sin[x(\sin\theta)] \sin m\theta \, d\theta$$

Or $$\pi J_n(x) = \int_0^{\pi} (\cos[x(\sin\theta)] \cos n\theta + \sin[x(\sin\theta)] \sin m\theta)\, d\theta$$

Or $$\pi J_n(x) = \int_0^{\pi} \cos(x(\sin\theta) - n\theta)\, d\theta$$

Or $$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos\left(n\theta - x(\sin\theta)\right) d\theta$$

7. STURM-LIOUVILLE FORM OF SECOND ORDER LINEAR DIFFERENTIAL EQUATION

The Sturm Liouville form of differential equation is given by

$$\Omega y(x) + \lambda\, \omega(x)\, y(x) = 0 \qquad ...(7.1)$$

where Ω is second order linear differential operator given as

$$\Omega = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \qquad ...(7.2)$$

here $p(x)$ and $q(x)$ are positive continuous functions and $p'(x)$ is also continuous in the given domain of the function. The operator Ω defined by equation (7.2) is also known as the self adjoint operator. Any second order linear differential equation could be written in the form of equation (7.1). To prove it consider the operator of equation (2.1) as

$$\Omega' = P_0(x)\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_2(x) \qquad ...(7.3)$$

Multiply equation (7.3) with a factor $\phi(x)$ given by

$$\phi(x) = \frac{1}{P_0(x)} \exp\int \frac{P_1(x)}{P_0(x)} dx$$

so that eqn (7.3) could be rewritten as

$$\phi(x)\Omega' = \exp\int \frac{P_1(x)}{P_0(x)} dx \frac{d^2}{dx^2} + \frac{P_1(x)}{P_0(x)} \exp\int \frac{P_1(x)}{P_0(x)} dx \frac{d}{dx} + \frac{P_2(x)}{P_0(x)}$$

$$\exp\int \frac{P_1(x)}{P_0(x)} dx$$

$$= \text{using } \frac{d}{dx}(ae^x) = ae^x \text{ one gets}$$

$$= \exp\int \frac{P_1(x)}{P_0(x)} dx \frac{d^2}{dx^2} + \frac{d}{dx}\left[\exp\int \frac{P_1(x)}{P_0(x)} dx\right] \frac{d}{dx} + \frac{P_2(x)}{P_0(x)}$$

$$\exp\int \frac{P_1(x)}{P_0(x)} dx$$

$$= \frac{d}{dx}\left[\exp\int \frac{P_1(x)}{P_0(x)} dx \frac{d}{dx}\right] + \frac{P_2(x)}{P_0(x)} \exp\int \frac{P_1(x)}{P_0(x)} dx$$

so that $\phi(x)\,\Omega' = \Omega$ with $p(x) = \exp\int\left(\frac{P_1(x)}{P_0(x)}\right) dx$ and

$$q(x) = \frac{P_2(x)}{P_0(x)} \exp\int \frac{P_1(x)}{P_0(x)} dx$$

Example 7.1. **Write Bessel's differential equation in sturm liouville form.**

Solution : Bessel's differential equation is given as

$$x^2 \frac{d^2y}{dx} + x\frac{dy}{dx} + (x^2 - m^2)\, y = 0$$

with $P_0(x) = x^2,\ P_1(x) = x,\ P_2(x) = -m^2$

so that $\exp\int \frac{P_1(x)}{P_0(x)} dx = \exp\int \frac{x}{x^2} dx = \exp\int \frac{1}{x} dx = \exp(\ln x) = x$

Multiply the Bessel's equation with $\frac{1}{P_2(x)} \exp\int \frac{P_1(x)}{P_2(x)} dx$

$$= \frac{x^2}{x} = x \quad \text{one gets}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(x - \frac{m^2}{x}\right) y = 0$$

or $$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{m^2}{x}y + xy = 0$$

with $p(x) = x$, $q(x) = \frac{-m^2}{x}$, $\omega(x) = x$ and $\lambda = 1$, as the parameters of sturm liouville form of differential equation.

Example 7.2. Write Hermite differential equation is sturm liouville form of differential equation.

Solution : Hermite's differential equation is given as

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2xy = 0$$

with $$P_0(x) = 1,\ P_1(x) = -2x \text{ and } P_2(x) = 2n$$

so that the multiplying factor $\frac{1}{P_2(x)}\exp\int\frac{P_1(x)}{P_0(x)}dx$ could be found as

$$\exp\int -2x\,dx = \exp\left(\frac{-2x^2}{2}\right) = e^{-x^2}$$

Multiplying Hermite equation with e^{-x^2} one gets

$$e^{-x^2}\left[\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2xy\right] = 0$$

or $$\frac{d}{dx}\left(e^{-x^2}\frac{dy}{dx}\right) + 2xy = 0$$

with $p(x) = e^{-x^2}$, $q(x) = 0$, $\lambda = 2n$ and $\omega(x) = e^{-x^2}$ as the parameters of Sturm Liouville form of differential equation

SUMMARY

- Second order linear Differential Equations could be solved using power series or frobenius method depending upon the nature of $p(x)$ and $q(x)$. If $p(x)$ and $q(x)$ are analytic then the equations are solved using power series, however if $p(x)$ and $q(x)$ have regular singularity the equations are solved using frobenius method.
- Ordinary point of a differential equation is a point x_0, if $p(x)$ and $q(x)$ are finite at x_0 and in its neighbourhood.
- Singularity of a differential equation is a point x_0, where $p(x)$ and $q(x)$ fails to be analytic and becomes infinite. If $\lim_{x\to x_0}(x-x_0)\,p(x)$ and $\lim_{x\to 0}(x-x_0)^2\,q(x)$ are finite the singularity is known

as regular singularity, however if these limits are infinite, the singularity is called irregular singularity of the differential equation.

- The Rodrigue's formula solution of some of the differential equations useful in solving various problems of physics are :

Legendre's Polynomials $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$

Associated Legendre's Polynomials $P_l^m(x) = (-1)^m (1 - x^2)^{m/2} \left(\frac{d^m P_l(x)}{dx^m} \right)$

Hermite's Polynomials $H_\nu(x) = (-1)^\nu e^{x^2} \frac{d^\nu}{dx^\nu} e^{-x^2}$

Laguerre Polynomials $L_k(x) = (-1)^k e^x \frac{d^k}{dx^k} (x^k e^{-x})$

Bessel's Polynomials $J_n(x) = (-1)^n x^n \left(\frac{1}{x} \frac{d}{dx} \right)^n J_0(x)$

- The generating functions $g(x, t)$ of some of the special polynomials are given as :

Legendre's Polynomials $\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l$

Hermite's Polynomials $e^{2xt - t^2} = \sum_{\nu=0}^{\infty} \frac{H_\nu(x) t^\nu}{\nu!}$

Laguerre Polynomials $\frac{1}{1-t} e^{-xt/1-t} = \sum_{k=0}^{\infty} \frac{L_k(x) t^k}{k!}$

Bessel's Polynomials $e^{x/2\left(t - \frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} J_m(x)\, t^m$

- The orthogonality of some of the special functions are :

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx = \frac{2}{2l+1} \delta_{lm} \quad \text{with } \delta_{lm} = \begin{cases} 0 & \text{if } l \neq m \\ 1 & \text{if } l = m \end{cases}$$

$$\int_{-1}^{1} P_l^m(x)\, P_{l'}^m(x)\, dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{lm}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_\nu(x)\, H_{\nu'}(x)\, dx = \sqrt{\pi}\, 2^\nu\, \nu!\, \delta_{\nu\nu'}$$

$$\int_{0}^{\infty} e^{-x} L_k(x)\, L_l(x)\, dx = (k!)^2 \delta_{kl}$$

$$\int_{0}^{1} x J_m(\alpha x)\, J_m(\beta x) = \frac{[J_{m+1}(\alpha x)]^2}{2} \delta_{\alpha\beta}$$

QUESTIONS

MULTIPLE CHOICE QUESTIONS

1. **The general form of linear differential equation of second order**
   (*a*) $(D^2 + P(x) D + Q(x))\, y = 0$ (*b*) $(D^2 - P(x) + Q(x))\, y = R(x)$
   (*c*) $(D^2 + P(k) + Q(3))\, y = R(x)$ (*d*) $(D^2 - P(k, 9) + Q(k, 3))\, y = 0$

2. **If $y = e^{2t}$ is a solution to $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + ky = 0$, what is the value of k ?**
   (*a*) 1 (*b*) 4
   (*c*) 0 (*d*) 6

3. **A differential equation is considered to be ordinary if it has**
   (*a*) one dependent variable (*b*) more than one dependent variable
   (*c*) one independent variable (*d*) more than one independent variable

4. **For differential equations $\left(\frac{dy}{dx}\right)^2 + \cos x\, y = x^3$, select the order of the differential equation and Linearity**
   (*a*) 0 and linear (*b*) 1 and linear
   (*c*) 1 and non linear (*d*) 0 and linear

5. **The Differential equation $\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)\, y = 0$, with n as a positive integer, is**
   (*a*) Legendre's Differential Equation (*b*) Bessel's Differential Equation
   (*c*) Chebyshev's Differential Equation (*d*) Hermite's Differential Equation

6. **The differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$, with n as an integer, is**
   (*a*) Legendre's Differential Equation (*b*) Bessel's Differential Equation
   (*c*) Chebyshev's Differential Equation (*d*) Hermite's Differential Equation

7. The function $P_n(1)$ is given as

(*a*) Zero (*b*) –1

(*c*) $P_n(-1)$ (*d*) 1

8. The generating function for Legendre's function is given as

(*a*) $(1+2xt+t^2)^{\frac{1}{2}}$ (*b*) $(1+2xt+t^2)^{-\frac{1}{2}}$

(*c*) $(1-2xt+t^2)^{\frac{1}{2}}$ (*d*) $(1-2xt+t^2)^{-\frac{1}{2}}$

9. $P_n(x)$ is considered as

(*a*) Non terminating series (*b*) Terminating Series

(*c*) Oscillatory Series (*d*) None of these

10. $Q_n(x)$ is considered as

(*a*) Non terminating series (*b*) Terminating Series

(*c*) Oscillatory Series (*d*) None of these

11. All the eigen values of $P_n(x)$ are

(*a*) Imaginary (*b*) Real and lie between –1 to 1

(*c*) Real and lie between 0 to ∞ (*d*) Real and equal

12. The generating function for Bessel's function is given as

(*a*) $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ (*b*) $e^{-\frac{x}{2}\left(t-\frac{1}{t}\right)}$

(*c*) $e^{-\frac{x}{2}\left(t+\frac{1}{t}\right)}$ (*d*) $e^{\frac{x}{2}\left(t+\frac{1}{t}\right)}$

13. The Rodrigues formula for Legendre's polynomials is given as

(*a*) $P_n(x)=\frac{n!}{2^n}\frac{d^n}{dx^n}(x^2-1)^n$ (*b*) $P_n(x)=\frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^{n-1}$

(*c*) $P_n(x)=\frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$ (*d*) $P_n(x)=\frac{n!}{2^n}\frac{d^n}{dx^n}(x^2-1)^{n-1}$

14. The polynomial $2x^2 + x + 3$ in terms of Legendre's polynomials is given as

(*a*) $\frac{1}{3}(4P_2(x) - 3P_1(x) + 11P_0(x))$ (*b*) $\frac{1}{3}(4P_2(x) + 3P_1(x) - 11P_0(x))$

(*c*) $\frac{1}{3}(4P_2(x) + 3P_1(x) + 11P_0(x))$ (*d*) $\frac{1}{3}(4P_2(x) - 3P_1(x) - 11P_0(x))$

15. In the Legendre's polynomial $P_5(x)=\lambda\left(x^5-\frac{70}{63}x^3+\frac{15}{63}x\right)$, λ is given as

(*a*) $\frac{63}{2}$ (*b*) $\frac{63}{5}$

(*c*) $\frac{63}{8}$ (*d*) $\frac{63}{10}$

16. The degree of $\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = x^2 \log\left(\dfrac{d^2y}{dx^2}\right)$

(*a*) One (*b*) Two

(*c*) Four (*d*) Undefined

17. Let $P_n(x)$ be a polynomial of degree n with real coefficients defined in the interval $2 \le n \le 4$. If $\int_2^4 P_n(x)P_m(x)\,dx = \delta_{mn}$ **, then**

(*a*) $P_0(x) = \dfrac{1}{\sqrt{2}}$ and $P_1(x) = \sqrt{\dfrac{3}{2}}\,(-3-x)$

(*b*) $P_0(x) = \dfrac{1}{\sqrt{2}}$ and $P_1(x) = \sqrt{3}\,(3+x)$

(*c*) $P_0(x) = \dfrac{1}{2}$ and $P_1(x) = \sqrt{\dfrac{3}{2}}\,(3-x)$

(*d*) $P_0(x) = \dfrac{1}{\sqrt{2}}$ and $P_1(x) = \sqrt{\dfrac{3}{2}}\,(3-x)$

18. The points where the series solution of the Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \frac{3}{2}\left(\frac{3}{2}+1\right) y = 0$$

will diverge, are located at

(*a*) 0 and 1 (*b*) 0 and –1

(*c*) –1 and 1 (*d*) $\frac{3}{2}$ and $\frac{5}{2}$

19. Consider the Bessel equation with $n = 0$ $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} + y = 0$**, which one of the following statement is correct?**

(*a*) Equation has regular singular points at $x = 0$ and $x = \infty$.

(*b*) Equation has 2 linearly independent solutions that are entire

(*c*) Equation has an entire solution and second linearly independent solution singular at $x = 0$.

(*d*) Limit $x \to \infty$, taken along x-axis, exists for both the linearly independent solutions.

20. Given the recurrence relation for the Legendre polynomials

$$(2n + 1)\, xP_n(x) = (n + 1)\, P_{n+1}(x) + nP_{n-1}(x)$$

Which of the following integrals has a non zero value?

(*a*) $\int_{-1}^{1} x^2 P_n(x) P_{n+1}(x)\,dx$ (*b*) $\int_{-1}^{1} x P_n(x) P_{n+2}(x)\,dx$

(*c*) $\int_{-1}^{1} x[P_n(x)]^2\,dx$ (*d*) $\int_{-1}^{1} x^2 P_n(x) P_{n+2}(x)\,dx$

21. **The generating function $g(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$ for the Legendre polynomials $P_n(x)$ is $g(x, t) = (1 - 2xt + t^2)^{-\frac{1}{2}}$. The value of $P_3(-1)$ is**

(*a*) 5/2 (*b*) 3/2

(*c*) +1 (*d*) –1

22. **Given $g(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$, for $|t| < 1$, the value of $P_5(-1)$ is**

(*a*) 0.26 (*b*) 1

(*c*) 0.5 (*d*) –1

23. **Given that $\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2 + 2tx}$, the value of $H_4(0)$ is**

(*a*) 12 (*b*) 6

(*c*) 24 (*d*) –6

24. **In the function $P_n(x) e^{-x^2}$ of a real variable *x*, $P_n(x)$ is polynomial of degree *n*. The maximum number of extrema that this function have is**

(*a*) $n + 2$ (*b*) $n - 1$

(*c*) $n + 1$ (*d*) n

25. **The polynomial $f(x) = 1 + 5x + 3x^2$ is written as linear combination of the Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ as $f(x) = \sum_n C_n P_n(x)$. The value of C_0 is**

(*a*) ¼ (*b*) ½

(*c*) 2 (*d*) 4

26. **Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of the differential equation**

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + f(t)x = 0,$$

And let $w(t) = x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt}$, if $w(0) = 1$, then $w(1)$ is given by

(*a*) 1 (*b*) e^2

(*c*) $1/e$ (*d*) $1/e^2$

27. **The value of the integral $\int_0^1 x[J_1(x)]^2\, dx$ is equal to**

(*a*) $[J_2(1)]^2$ (*b*) 0

(*c*) $\frac{1}{2}[J_2(1)]^2$ (*d*) $[J_1(1)]^2$

SHORT ANSWER TYPE QUESTIONS

1. Discuss whether two Frobenius series solutions exist or do not exist for the following equations :

$$2x^2 y'' + x(x + 1)y' - (\cos x) y = 0$$

2. Given that $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx}$, Find the value of $H_4(0)$.

3. The Hermite Polynomial $H_n(x)$, satisfies the differential equation

$$\frac{d^2 H_n(x)}{dx^n} - 2x\frac{dH_n(x)}{dx} + 2nH_n(x) = 0.$$

Prove that the corresponding generating function

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = G(x,t)$$

satisfies the equation

$$\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2t\frac{\partial G}{\partial t} = \mathbf{0}$$

4. If the function $f(x)$ is defined by the integral equation

$$\int_0^x f(x)\,dx = xG(1, x)$$

Prove that It can be expressed as

$$\sum_{n,m=0}^{\infty} x^{n+m} P_n(1)\, P_m(1)$$

5. Prove that the second order derivative of the Hermite polynomial of nth order, *i.e.* $H_n''(n \geq 2)$ can be written as $4n(n-1)\,H_{n-2}(x)$.

6. Find the value of the integral $\int_{-1}^{1}(1-x^2)\,[P_n'(x)]^2\,dx$.

7. Prove that any function $f(x)$ which is finite and single valued in the interval $-1 \leq x \leq 1$, and which has a finite number of discontinuities within this interval can be expressed as a series of legendre polynomials.

8. Prove that

$$\int_{-1}^{1} x^2 P_{n+1}(x) P_{n-1}(x)\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

9. Evaluate $\int_{-1}^{1} x^2\,[P_n(x)]^2\,dx$

10. Prove that $P_{-(n+1)}(x) = P_n(x)$

LONG ANSWER TYPE QUESTIONS

1. Solve $x(x-1)y'' + (3x-1)y' + y = 0$

2. Find series solution of the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

3. Expand the function $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$ in terms of Legendre's polynomials.

4. Prove that any arbitrary function $f(x)$ could be represented in terms of Legendre's polynomials, subjected to the condition that $f(x)$ is defined from $x = -1$ to $x = 1$.

5. Express the function

$$f(x) = \begin{cases} 0, & -1 < x \le 0 \\ x & 0 < x < 1 \end{cases}$$

in terms of Legendre's Polynomials

6. Show that $\sqrt{\dfrac{\pi x}{2}}\, J_{-\frac{3}{2}}(x) = -\sin x - \dfrac{\cos x}{x}$

7. Prove that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

8. Prove that $x^2 J_n''(x) = (n^2 - n - x^2) J_n(x) + x J_{n+1}(x)$

9. Show that $H_n(x) = 2^{n+1} e^{x^2} \int_x^{\infty} e^{-t^2} t^{n+1} P_n\left(\dfrac{x}{t}\right) dt$

10. Using the generating function of Hermite polynomials evaluate the value of (*a*) $H_2(x)$, (*b*) $H_3(x)$

HINTS / ANSWERS

MULTIPLE CHOICE QUESTION

1. (*a*)	**2.** (*d*)	**3.** (*c*)	**4.** (*c*)	**5.** (*a*)	**6.** (*b*)	**7.** (*d*)	**8.** (*d*)	**9.** (*b*)	**10.** (*a*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*c*)	**16.** (*d*)	**17.** (*d*)	**18.** (*c*)	**19.** (*c*)	**20.** (*d*)
21. (*d*)	**22.** (*d*)	**23.** (*a*)	**24.** (*c*)	**25.** (*c*)	**26.** (*d*)	**27.** (*c*)			

2. **Hint :** Substituting $y = e^{2t}$ in $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + ky = 0$, one gets

$$4e^{2t} - 10e^{2t} + ke^{2t} = 0$$

Or

$$4 - 10 + k = 0$$

20. **Hint :** Short answer type question no. 8

21. **Hint :** $P_n(-1) = (-1)^n P_n(1) = (-1)^n.1 = (-1)^n$ and for $n = 3$, $P_3(-1) = -1$

22. **Hint :** $P_n(-1) = (-1)^n P_n(1) = (-1)^n.1 = (-1)^n$ and for $n = 5$, $P_5(-5) = -1$

23. **Hint :** $\sum_{n=0}^{\infty} H_n(x)\dfrac{t^n}{n!} = e^{-t^2+2tx} \Rightarrow \sum_{n=0}^{\infty} H_n(0)\dfrac{t^n}{n!} = e^{-t^2} = 1 - t^2 + \dfrac{t^4}{2!} - \dfrac{t^6}{3!}$

Comparing the coefficients of t^4 one gets

$$H_4(0)\frac{t^4}{4!} = \frac{1}{2!} \Rightarrow H_4(0) = \frac{4!}{2!} = 12$$

25. Hint : $f(x) = 1 + 5x + 3x^2$

$$3x^2 = 2P_2(x) + 1$$

Thus $f(x) = 2P_2(x) + 1 + 5P_1(x) + 1 = 2P_2(x) + 5P_1(x) + 2P_0(x)$

26. Hint : $w = e^{-\int P(t)\,dt}$ here $P(t) = 2$, so that $w = e^{-\int 2dt} = e^{-2t}$, thus $w(1) = e^{-2}$

27. Hint : Using orthogonality relation of Bessel equation one have $\int_0^1 x[J_n(\alpha x)]^2 dx = \frac{1}{2}[J_{n+1}(\alpha)]^2$

SHORT ANSWER QUESTIONS

1. The equation can be rewritten as

$$y'' + \frac{x(x+1)}{2x^2}y' - \frac{\cos x}{2x^2}y = 0$$

Hence $x = 0$ is a singular point to check its singularity, consider

$$\lim_{x\to 0}(x-0)\frac{x(x+1)}{2x^2} = \lim_{x\to 0}\frac{x+1}{2} = \frac{1}{2}$$

Similarly

$$\lim_{x\to 0}(x-0)^2\frac{\cos x}{2x^2} = \lim_{x\to 0}\cos x = 1$$

Both are finite hence Frobenius series solutions exist.

2. $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx} \Rightarrow \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!} = e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!}$

Comparing the coefficients of t^4 one gets

$$H_4(0)\frac{t^4}{4!} = \frac{1}{2!} \Rightarrow H_4(0) = \frac{4!}{2!} = 12$$

3. $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = G(x,t)$

Differentiating the above equation *w.r.t.* x one gets

$$\sum_{n=0}^{\infty} H'_n(x)\frac{t^n}{n!} = G'(x,t), \quad \sum_{n=0}^{\infty} H''_n(x)\frac{t^n}{n!} = G''(x,t)$$

Also differentiating *w.r.t* 't', one gets

$$\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} nH_n(x)\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

Multiplying the above equation with t one gets

$$t\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} nH_n(x)\frac{t^n}{n!}$$

Substituting these values in Hermite's equation one gets

$$\frac{\partial^2 G}{\partial x^2} - 2x\frac{\partial G}{\partial x} + 2t\frac{\partial G}{\partial t} = 0$$

4. $$G(x,1) = (1-2x+x^2)^{-\frac{1}{2}} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n P_n(1)$$

Also

$$\int_0^x f(x)\,dx = x\,G(1,x) = \frac{x}{1-x}$$

Differentiating both sides one gets

$$f(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} x^n P_n(1).\sum_{m=0}^{\infty} x^m P_m(1)$$

5. The generating Function for Hermite polynomial $e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$

Considering the second derivative *w.r.t.* x one gets

$$4t^2 e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n''(x)\frac{t^n}{n!}$$

or $$4\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+2}}{n!} = \sum_{n=0}^{\infty} H_n''(x)\frac{t^n}{n!}$$

Comparing the coefficients of t^n on both sides, one gets

$$H_n''(x) = 4n(n-1)\,H_{n-2}(x)$$

6. Using orthogonality relation of Legendre polynomials one have $\int_0^1 [P_n(x)]^2\,dx = \frac{2}{2n+1}$

And also $\frac{d}{dx}[(1-x^2)P_n'(x)] = -n(n-1)\,P_n(x)$

Multiply the two sides with $P_n(x)$ and integrating the two sides *w.r.t.* x between the limits –1 to 1 one gets

$$\int_{-1}^{1} \frac{d}{dx}[(1-x^2)P_n'(x)]P_n(x)\,dx = -\int_{-1}^{1} n(n+1)\,P_n(x)\,P_n(x)\,dx$$

$$-\int_{-1}^{1} [(1-x^2)][P_n'(x)]^2\,dx = -n(n+1)\int_{-1}^{1} [P_n(x)]^2\,dx = -\frac{2n(n+1)}{2n+1}$$

7. Let $$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + ... = \sum_{n=0}^{\infty} A_n P_n(x)$$

Multiplying both sides by $P_m(x)\,dx$ and integrating with respect to x from $x = -1$ to $x = 1$ gives

$$\int_{-1}^{1} f(x)\,P_m(x)\,dx = \sum_{n=0}^{\infty} A_n \int_{-1}^{1} P_n(x)P_m(x)\,dx$$

By means of the orthogonality property of the Legendre polynomials one can write

$$\sum_{n=0}^{\infty} A_n \frac{2\delta_{nm}}{2n+1} = \int_{-1}^{1} f(x) P_m(x)\, dx$$

Or
$$A_m \frac{2}{2m+1} = \int_{-1}^{1} f(x) P_m(x)\, dx$$

Or
$$A_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

8. In the recurrence relation $(2n+1)\, xP_n(x) = (n+1)\, P_{n+1}(x) + nP_{n-1}(x)$

Replacing n by $n+1$ and $n-1$, respectively one gets

$$(2n+3)\, x\, P_{n+1}(x) = (n+2)\, P_{n+2}(x) + (n+1)\, P_n(x) \qquad ...(1)$$

And

$$(2n-1)\, xP_{n-1}(x) = nP_n(x) + (n-1)\, P_{n-2}(x) \qquad ...(2)$$

Considering the product of equation (1) and (2) and integrating between the limits –1 to 1, one gets

$$(2n+3)(2n-1) \int_{-1}^{+1} x^2 P_{n-1}(x) P_{n-1}(x)\, dx = n(n+2) \int_{-1}^{1} P_n(x) P_{n+2}(x)\, dx +$$

$$n(n+1) \int_{-1}^{1} [P_n(x)]^2\, dx + (n-1)(n+2) \int_{-1}^{1} P_{n-2}(x) P_{n+2}(x)\, dx + (n^2-1) \int_{-1}^{1} P_n(x)\, P_{n-2}(x)\, dx$$

Using orthogonality relation of Legendre polynomials one gets

$$(2n+3)(2n-1) \int_{-1}^{+1} x^2 P_{n+1}(x) P_{n-1}(x)\, dx = \frac{2n(n+1)}{2n+1}$$

$$\int_{-1}^{+1} x^2 P_{n+1}(x) P_{n-1}(x)\, dx = \frac{2n(n+1)}{(2n+1)(2n+3)(2n-1)}$$

9. Squaring the recurrence relation $(2n+1)\, xP_n(x) = (n+1)\, P_{n+1}(x) + nP_{n-1}(x)$ and integrating between the limits –1 to 1, one gets

$$(2n+1)^2 \int_{-1}^{+1} x^2 [P_n(x)]^2\, dx = (n+1)^2 \int_{-1}^{1} [P_{n+1}(x)]^2\, dx +$$

$$2n(n+1) \int_{-1}^{1} P_{n+1}(x) P_{n-1}(x)\, dx + n^2 \int_{-1}^{1} [P_{n-1}(x)]^2\, dx$$

Using orthogonality relation of Legendre polynomials one gets

$$(2n+1)^2 \int_{-1}^{+1} x^2 [P_n(x)]^2\, dx = \frac{2(n+1)^2}{2n+3} + \frac{2n^2}{2n-1}$$

$$(2n+1)^2 \int_{-1}^{+1} x^2 [P_n(x)]^2 dx = \frac{2(n+1)^2}{2n+3} + \frac{2n^2}{2n-1}$$

$$\int_{-1}^{+1} x^2 [P_n(x)]^2 dx = \frac{2(n+1)^2}{(2n+3)(2n+1)^2} + \frac{2n^2}{(2n-1)(2n+1)^2}$$

10. From Laplace first integral, one knows

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x \pm \sqrt{(x^2-1)} \cos\varphi \right]^n d\varphi$$

Replacing n with $-(n+1)$ one gets

$$P_{-(n+1)}(x) = \frac{1}{\pi} \int_0^{\pi} \left[x \pm \sqrt{(x^2-1)} \cos\varphi \right]^{-(n+1)} d\varphi$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{\left[x \pm \sqrt{(x^2-1)} \cos\varphi \right]^{(n+1)}} d\varphi$$

Which is the representation of legendre polynomial as per the Laplace second representation, *i.e.*

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{1}{\left[x \pm \sqrt{(x^2-1)} \cos\varphi \right]^{(n+1)}} d\varphi$$

Hence $P_{-(n+1)}(x) = P_n(x)$

LONG ANSWER QUESTIONS

1. **Hint :** Solve using Frobenius method and the value of $\alpha = 0$ the one solution will be

$$y = a_0 (1 + x + x^2 + \ldots\ldots\ldots x^4) = \sum x^n a_0$$

The second solution is given by

$$\left(\frac{\partial y_1}{\partial m}\right)_{m=0} = \frac{\partial}{\partial m}\left(a_0' x^m \sum_{n=0}^{\infty} x^n \right)$$

$$\left. \left(a_0 x^m \ln x \sum_{n=0}^{\infty} x^n \right) \right|_{m=0} = a_0 \ln x \sum_{n=0}^{\infty} x^n$$

which is the second solution so that

$$\boldsymbol{y = a_0 \left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \ln x \; x^n \right]}$$

2. **Hint :** $P(x) = \frac{1}{x}$ and $Q(x) = \frac{-1}{x}$, thus equation could be solved using Frobenius method with $\alpha = 0$,

thus similar method as adopted for problem could be applied to solve this equation.

Ans. $$y = a_0 \sum \frac{1}{(n!)^2} x^n$$

3. **Hint:** $$f(x) = \sum_{n=0}^{\infty} a_l P_l(x)\, dx, \text{ then}$$

$$a_l = \left(l+\frac{1}{2}\right) \int_{-1}^{+1} f(x) P_l(x)\, dx$$

$$= \left(l+\frac{1}{2}\right)\left[\int_{-1}^{0} 0\, P_l(x)\, dx + \int_{0}^{1} P_l(x)\, dx\right] = \left(l+\frac{1}{2}\right)\int_{0}^{1} P_l(x)\, dx$$

$$a_0 = \frac{1}{2}\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 1.dx = \frac{1}{2}(x)\Big|_0^1 = \frac{1}{2}$$

$$a_1 = \frac{3}{2}\int_0^1 P_l(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{2}\frac{x^2}{2}\Big|_0^1 = \frac{3}{4}$$

So that $$\boldsymbol{f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) + ...}$$

4. **Hint :** Consider that the function $f(x)$ defined from $x = -1$ to $x = 1$ could be represented as

$$f(x) = a_1P_1(x) + a_2P_2(x) + a_3 P_3(x) \text{ .... } + a_l P_l(x)$$

Multiply the above eqn. with $P_n(x)$ and integrate with respect to x between the limits $x = -1$ to $x = 1$, one gets

$$\int_{-1}^{+1} f(x) P_n(x)\, dx = a_1 \int_{-1}^{+1} P_1(x) P_n(x)\, dx + a_2 \int_{-1}^{+1} P_2(x) P_n(x)\,dx$$

$$+ .... + a_l \int_{-1}^{+1} P_l(x) P_n(x)\,dx + ... + a_n \int_{-1}^{+1} P_n(x) P_n(x)\,dx$$

Using the orthogonality relation of Legendre's polynomials given by eqn. (3.16) one gets

$$\int_{-1}^{+1} f(x) P_n(x)\,dx = a_1(0) + a_2(0) + ...a_n\cdot\frac{2}{2n+1} + ...a_l(0)$$

Or $$a_n = \frac{2n+1}{2}\int_{-1}^{+1} f(x) P_n(x)\, dx$$

5. **Hint :** Solve it as Problem 3.

Ans. $$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + ...$$

6. **Hint :** Considering $$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!}\left(\frac{x}{2}\right)^{2n+m}$$

rearranging the terms one gets

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!}\left(\frac{x}{2}\right)^{2n}$$

Let $m = -\frac{3}{2}$, one gets

$$J_{-\frac{3}{2}}(x) = \left(\frac{x}{2}\right)^{-3/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\left(n-\frac{3}{2}\right)!}\left(\frac{x}{2}\right)^{2n}$$

Here $$\left(n-\frac{3}{2}\right)! = \Gamma\left(n-\frac{3}{2}+1\right) = \Gamma\left(n-\frac{1}{2}\right)$$

and $$\Gamma\left(n-\frac{1}{2}\right) = \frac{(2n-3)!!\sqrt{\pi}}{2^{n-1}}$$

Hence $$J_{-\frac{3}{2}}(x) = \left(\frac{x}{2}\right)^{-3/2} \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{n-1}}{n!\,(2n-3)\,!!\sqrt{\pi}}\left(\frac{x}{2}\right)^{2n}$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\frac{2}{x} \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{n-1}}{n!(2n-3)!!}\left(\frac{x}{2}\right)^{2n}$$

$$= \sqrt{\frac{2}{\pi x}}\,\frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!!\,(2n-3)\,!!}x^{2n}$$

$$= \sqrt{\frac{2}{\pi x}}\,\frac{1}{x} \sum_{n=0}^{\infty}\left[\frac{(-1)^n\,(2n-1)}{(2n)!}x^{2n}\right]$$

$$= \sqrt{\frac{2}{\pi x}}\,\frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{(2n-1)!} - \frac{1}{(2n)!}\right] x^{2n}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n-1}}{(2n-1)!} - \frac{1}{x}\,\frac{x^{2n}}{(2n)!}\right]$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[-\sin x - \frac{\cos x}{x}\right]$$

7. Hint :

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+m)!}\left(\frac{x}{2}\right)^{2n+m}$$

For

$$m = \frac{1}{2}$$

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\left(n+\frac{1}{2}\right)!}\left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{n+1}}{n!\,(2n+1)!!\sqrt{\pi}}\left(\frac{x}{2}\right)^{2n}$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n\, 2}{(2n)!\,(2n+1)!!\sqrt{\pi}}\, x^{2n}$$

$$= \sqrt{\frac{2}{x\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n-1)!}$$

$$= \sqrt{\frac{2}{x\pi}} \sin x$$

8. Hint : Consider the Bessel's differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

and the solution is $J_n(x)$ such that the above eqn. could be rewritten as

$$x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2) J_n(x) = 0 \qquad ...(1)$$

Also $$xJ_n'(x) = n\, J_n(x) - x\, J_{n+1}(x) \qquad ...(2)$$

Substituting (2) in (1), one gets

$$x^2 J_n''(x) + n\, J_n(x) - x\, J_{n+1}(x) + (x^2 - n^2) J_n(x) = 0$$

Or $$x^2 J_n''(x) - x\, J_{n+1}(x) - (n^2 - x^2 - n) J_n(x) = 0$$

Or $$x^2 J_n''(x) = (n^2 - x^2 - n)\, J_n(x) + x\, J_{n+1}(x)$$

10. $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^2 - 12x$