UNIT 4 PHYSICAL PRINCIPLES

Structure

4.1 INTRODUCTION

So far in this block we have provided the basic information which is useful in astronomy. You have learnt about astronomical quantities of interest, various coordinate systems, astronomical instruments and techniques. We now turn our attention to astrophysics. The aim of astrophysics is to apply principles of physics to understand and explain the behaviour of various astronomical systems.

There are certain physical principles and concepts which are used in astrophysics so universally that it is worthwhile to discuss them before we begin studying specific astronomical systems. We have decided to put together some such basic physical principles in this Unit. You must have already learnt many of these principles in your other physics courses. Now you will learn how these can be applied to astrophysical systems.

Let us consider some issues which are of universal concern in astrophysics. We know that **gravitation** is the dominant force in virtually any astrophysical setting. Since gravitation is always attractive, it must be balanced in some way in a system which is not shrinking. We shall discuss a very general and powerful principle called the **virial theorem**, which helps us understand how gravitation is balanced in astrophysical systems.

Another topic of importance in astrophysics is the **interaction of radiation with matter**. Astrophysics is a very special science in which we cannot do experiments with our systems (stars, galaxies, etc.) in our laboratories. Virtually everything we know about these systems is learnt by analysing the radiation reaching us from these systems. If we want to make inferences about the systems which emitted the radiation or through which the radiation passed, then we need to understand how matter and radiation interact with each other. Many astronomical systems like stars emit radiation simply because they are hot. So we often need to apply various principles of thermal physics to understand how matter in these systems behaves. Therefore, we plan to recapitulate some of the important results of systems in **thermodynamic equilibrium** and then develop the theory of transfer of radiation through matter.

Objectives

After studying this unit, you should be able to:

- apply virial theorem to simple astrophysical systems;

- identify the situations in astrophysics to which Newton's theory of gravitation or general theory of relativity can be applied;
- determine the specific intensity, energy density, radiant flux and radiation, pressure for a given radiation field; and
- solve the radiative transfer equation for simple cases and interpret the results.

Study Guide

In this unit, we will be using certain concepts discussed in various units of the physics electives PHE-01 entitled 'Elementary Mechanics' (Unit 10), PHE-06 entitled 'Thermodynamics and Statistical Mechanics (Unit 9), and PHE-11 entitled 'Modern Physics' (Unit 9). Please keep these units handy for ready reference.

4.2 GRAVITATION IN ASTROPHYSICS

In your elementary physics courses, you must have learnt about two important long-range forces, whose range of influence extends to infinity. These are the **gravitational** force and the **electromagnetic** force. The electromagnetic force can be attractive or repulsive depending on the nature of charges. So, if a system has equal amounts of positive and negative charges, and if there are no relative motions between these two types of charges, then electromagnetic forces are screened off. That is, the system does not produce a large-scale electromagnetic field. On the other hand, gravitation is always attractive and cannot be screened off. So it is the dominant force acting over the entire universe.

Since gravitation is always attractive, a natural question to ask is: Why do the celestial objects not shrink in size? Obviously, it has to be balanced in a system which is not shrinking in size (such as a star or a galaxy). We shall now use Newtonian theory of gravitation to discuss an important theorem (**virial theorem**) which tells us how this balancing takes place. Then we shall briefly consider whether Newtonian theory of gravitation is adequate in astrophysics or whether we have to apply a more complete theory of gravitation – the general theory of relativity due to Einstein.

4.2.1 Virial Theorem

Why is our solar system not shrinking in size? We shall analyse this problem in the **non-inertial frame of reference attached to a planet**. From Unit 10 of the course PHE-01 entitled 'Elementary Mechanics', you know the answer: the Sun's gravitational attraction on a planet is balanced by the centrifugal force due to the orbital motion of the planet. Let M and m be the masses of the Sun and a planet in the solar system. For simplicity, if we assume the planet to go in a circular orbit of radius r with speed v, then the force balance equation in the given non-inertial frame of reference is

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

We can rewrite this equation in the form

$$2\left(\frac{1}{2}mv^2\right) - \frac{GMm}{r} = 0 \tag{4.1}$$

Now note that the gravitational potential energy of the system (the Sun and the planet) is

$$E_G = -\frac{GMm}{r},$$

whereas the kinetic energy of the system is

$$E_K = \frac{1}{2} m v^2$$

if the Sun is assumed to be at rest. We can now rewrite Eq. (4.1) in the following form

$$2E_K + E_G = 0 \qquad (4.2)$$

This is the **virial theorem** for a planet going around the Sun.

What is the significance of this result? This tells us that the gravitational potential energy and the kinetic energy of a system will have to be of the same order if gravitation is to be balanced by motion. We have proved virial theorem for the simple case of a planet going around the Sun in a circular orbit. However, Eq. (4.2) can be proved quite generally for a system in which gravitation is balanced by motions such that the system is not shrinking in size. The motions needs not be circular, but can be of any type. For example, inside a star, gravity is balanced by thermal motions of its particles (atoms, electrons, ions). Even in this situation, Eq. (4.2) can be shown to hold provided we take the total kinetic energy of all the particles in the star for E_K. If a galaxy or a star cluster is not shrinking in size, the total kinetic energy E_K of the stars in it should be related to the total gravitational potential energy E_G by Eq. (4.2).

Fig.4.1: Spiral galaxies

In a type of galaxy known as spiral galaxy, stars seem to be moving in nearly circular orbits. However, in typical star clusters and in galaxies known as elliptical galaxies, stars move randomly. Due to these random motions, the stars do not fall to the centre due to gravitation. Do you feel puzzled by the idea that random motions can balance gravitation? To understand this concept, consider the air around you. Earth's gravity is pulling all the molecules of the air. Then why are not all molecules settling on the floor of the room due to this attraction? It is the random motion of the molecules which prevents this from happening.

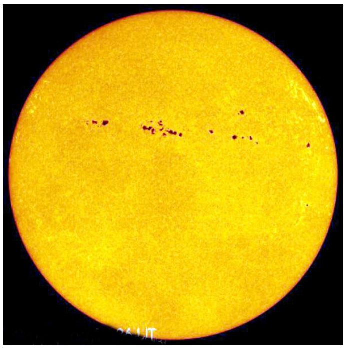

Fig.4.2: The Sun

Deriving the virial theorem Eq. (4.2) for a completely general situation is a very mathematically involved problem. It is beyond the scope of this elementary course. So let us apply the virial theorem in a simple situation so that you feel comfortable with it.

Example 1: Estimating the average temperature in stellar interior

A star (we have the Sun in mind) has a mass of 10^{33}g and a radius of 10^{11}cm. Make an order-of-magnitude estimate of the average temperature in the interior of the star.

Solution

Since this is an order of magnitude estimate, we shall take the virial theorem to imply that the potential and the kinetic energies are approximately equal. In c.g.s. units, we take the following approximate values of various physical constants: gravitational constant $G \approx 10^{-7}$, Boltzmann constant $k_B \approx 10^{-16}$, mass of hydrogen atom $m_H \approx 10^{-24}$.

The gravitational potential at the surface of the star has the magnitude *GM/R*. The approximate gravitational potential energy of the star is given by multiplying this by *M*, which is

$$\frac{GM^2}{R} = \frac{10^{-7} \times \left(10^{33}\right)^2}{10^{11}} \approx 10^{48} \text{erg}$$

Now we need to equal the total kinetic energy to this. Now, suppose that the star is made up of hydrogen. Then, the star consists of about M/m_H particles. Each of them has kinetic energy of order k_BT. Therefore, the total kinetic energy of the star is

$$\frac{M}{m_H} k_B T \approx \frac{10^{33} \times 10^{-16} T}{10^{-24}} \approx 10^{41} T$$

If this is equated to 10^{48} erg, then we obtain the temperature of the star as

$$T \approx 10^7 \text{ K}$$

Note that detailed calculations suggest a temperature of about 15 million degrees at the centre of the Sun. The above order of magnitude estimate thus provides a fairly good estimate of the average temperature inside a star.

You may now like to solve a problem to fix these ideas.

SAQ 1

Spend 10 min.

A globular cluster of stars has about a million stars. The stars inside such a cluster of radius 10^{20} cm have random velocities of order 10^6 cm s^{-1}. Estimate the mass of the star cluster and the number of stars in it. Take the mass of a star to be about 10^{33}g.

Hint: The mass of the cluster is *Nm*, where *N* is the number of stars in the cluster. Now, equate the total K.E. of the cluster to its P.E.

After Newton formulated his theory of universal gravitation, for more than two centuries it was regarded as a supreme example of a successful physical theory. However, in 1915, Einstein showed that this theory was incomplete and formulated his new theory of gravitation known as general relativity. In this section, we investigate the situations in which Newton's theory may have to be replaced by the general theory of relativity.

4.2.2 Newton versus Einstein

We now know that Newton's theory is only an approximation. But it is such an exceptionally good approximation in most circumstances that we do not need general relativity at all. Only **when the gravitational field is sufficiently strong, we have to apply general relativity**. Although we shall not discuss general relativity in this elementary course, we would like to point out when you can safely use Newtonian theory and when general relativity is needed.

Even people without any technical knowledge of general relativity now-a-days have heard of black holes. These are objects with gravitational fields so strong that even light cannot escape. Let us try to find out when this happens. Newtonian theory does not tell us how to calculate the effect of gravitation on light. So let us figure out when a particle moving with speed of light *c* will get trapped, according to Newtonian theory. Suppose we have a spherical mass *M* of radius *r* and a particle of mass *m* is ejected from its surface with speed c. The gravitational potential energy of the particle is

$$-\frac{GMm}{r}$$

If we use the non-relativistic expression for kinetic energy for a crude estimate (we should actually use special relativity for a particle moving with c!), then the total energy of the particle is

$$E = \frac{1}{2}mc^2 - \frac{GMm}{r}$$

Newtonian theory tells us that the particle will escape from the gravitational field if E is positive and will get trapped if E is negative. In other words, the condition of trapping is

$$\frac{1}{2}mc^2 - \frac{GMm}{r} < 0$$

or

$$\frac{2GM}{c^2 r} > 1 \qquad (4.3)$$

It turns out that more accurate calculations using general relativity give exactly the same condition (4.3) for light trapping, which we have obtained here by crude assumptions.

***General relativity is needed* when the factor**

$$f = \frac{2GM}{c^2 r} \qquad (4.4)$$

is of the order unity.

***Newtonian theory is quite adequate* if *f* is much smaller than 1.**

Let us investigate the case of the Sun.

Example 2

Let us examine if the Newtonian theory is adequate for the Sun.

Solution

The Sun has mass 1.99×10^{33} g and radius 6.96×10^{10} cm. Substituting these values in Eq. (4.4), we get

$$f = 4.24 \times 10^{-6} << 1$$

Hence, Newtonian theory is quite adequate for all phenomena in the solar system. Only if we want to calculate very accurate orbits of planets close to the Sun (such as Mercury), we have to take into consideration general relativity.

When is general relativity applicable in the case of the Sun? We can use Eq. (4.3) to calculate the radius to which the solar mass has to be shrunk such that f is of order unity. Then the light emitted at its surface gets trapped. Why don't you do this calculation yourself?

SAQ 2

Spend 2 min.

Calculate r for the Sun such that $f \sim 1$.

You would have calculated the radius to be 3 km. Therefore, general relativity will apply to the Sun, once it shrinks to this size. As we shall discuss in more detail in Block 3, when the energy source of a star is exhausted, the star can collapse to very compact configurations like neutron stars or black holes. General relativity is needed to study such objects.

If matter is distributed uniformly with density ρ inside radius r, then we can write

$$M = \frac{4}{3}\pi r^3 \rho$$

and Eq. (4.4) becomes

$$f = \frac{8\pi}{3}\frac{Gr^2\rho}{c^2} \tag{4.5}$$

We note that f is large when ρ is large or r is large (for given ρ). The density ρ is very high inside objects like neutron stars. You may ask: Can there be situations where general relativity is important due to large r? We know of one object with very large size – our universe itself. The distance to farthest galaxies is of the order 10^{28} cm. It is very difficult to estimate the average density of the universe accurately. Probably it is of the order 10^{-29} g cm^{-3}. You may like to substitute these values in Eq. (4.5), and calculate f.

SAQ 3

Spend 2 min.

Calculate the value of f from Eq. (4.5) using the data given above for the Universe.

The result of SAQ 3 tells us that we should use general relativity to study the dynamics of the whole universe. This subject is called **cosmology**.

Thus, in astrophysics, **we have two clear situations in which general relativity is very important**:

- the study of collapsed stars and
- the study of the whole universe (or cosmology).

In most other circumstances, we can get good results by applying Newtonian theory of gravitation.

As we mentioned in the introduction, basic principles of thermal physics apply to many astronomical systems with high temperatures. Let us briefly revisit these principles.

4.3 SYSTEMS IN THERMODYNAMIC EQUILIBRIUM

You have learnt about thermodynamic equilibrium in the course PHE-06 entitled Thermodynamics and Statistical Mechanics. If a system is in thermodynamic equilibrium, then several important principles of physics can be applied to that system. Let us first recapitulate some important laws and equations, relevant for such systems, namely, Maxwellian velocity distribution, Boltzmann distribution law, Saha's equation and Planck's law of black body radiation. Afterwards, we shall discuss whether we can assume astrophysical systems to be in thermodynamic equilibrium and whether these principles can be applied to them.

Maxwellian velocity distribution

You have studied about Maxwellian velocity distribution in Unit 9 of PHE-06. Different particles in a gas move around with different velocities. Recall that if the gas is in thermodynamic equilibrium at temperature *T*, the number of particles per unit volume having speeds between *v* and *v* + *dv* is given by

$$dN_v = 4\pi N\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right)dv \tag{4.6}$$

where *N* is the number of particles per unit volume and *m* is the mass of each particle.

Boltzmann and Saha Equations

You have studied in Unit 9 of the physics elective PHE-11 entitled Modern Physics that a hydrogen atom has several different energy levels. It is also possible to break the hydrogen atom into a proton and an electron. This process of removing an electron from the atom is called **ionisation**. If a gas of hydrogen atoms is kept in thermodynamic equilibrium, then we shall find that a certain fraction of the atoms will occupy a particular energy state and also a certain fraction will be ionised. The same considerations hold for other gases.

If N_0 is the number density of atoms in the ground state, then the number density N_e of atoms in an excited state with energy *E* above the ground state is given by

$$\frac{N_e}{N_0} \propto \exp\left(-\frac{E}{k_B T}\right) \tag{4.7}$$

This is the **Boltzmann distribution law**.

In 1919, the famous Indian physicist M.N. Saha derived an equation which tells us what fraction of a gas will be ionised at a certain temperature *T* and pressure *p*. The derivation of this equation involves some statistical mechanics. Here we merely quote the result without derivation. If n_{I} is the number of hydrogen atoms out of which n_{II} are ionised at temperature *T*, then Saha's equation gives

$$\frac{n_{\mathrm{II}}}{n_{\mathrm{I}}} p_e = \left(\frac{2\pi m_e}{h^2}\right)^{3/2} (k_B T)^{5/2} \exp\left[-\frac{I}{k_B T}\right] \tag{4.8}$$

where *I* is the ionisation potential of hydrogen, p_e is the partial pressure of electrons, *h* is Planck's constant and m_e is the mass of electron. A form convenient for calculation is

$$\log\left(n_{\mathrm{II}}\, p_e / n_{\mathrm{I}}\right) = 2.5\log T - (5040/T)\,\mathrm{I} - 0.48$$

Spend 5 min.

SAQ 4

Assuming p_e to be 100 dyne/cm^2, calculate the fraction of hydrogen atoms ionised at *T* = 10,000 *K*. The ionisation potential of hydrogen is 13.6 eV.

Planck's law of blackbody radiation

You have studied this law and its consequences in Unit 15 of PHE-06. You know that when radiation is in thermodynamic equilibrium with matter, it is called blackbody radiation. The spectral distribution of energy in blackbody radiation is given by the famous law derived by Planck in 1900 (see Fig. 4.3). The energy density $u_\nu d\nu$ lying in the frequency range between ν and $\nu + d\nu$ is given by

$$u_\nu d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T}\right)-1} \quad (4.9)$$

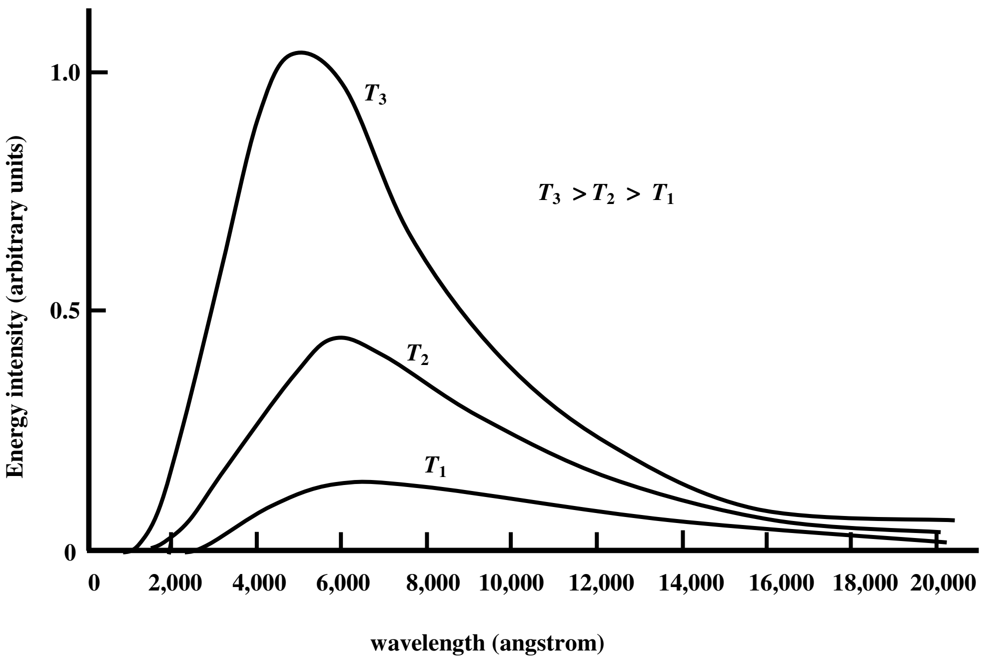

Fig.4.3: Blackbody radiation curve

We can now use these results to understand the interaction of matter with radiation.

4.4 THEORY OF RADIATIVE TRANSFER

Matter can both emit and absorb radiation. It is possible to use quantum mechanics to calculate the rates at which atoms emit or absorb energy. Here, however, we shall not do it. We would study the processes of emission and absorption by matter, by introducing suitable coefficients of emission and absorption. **Radiative transfer** is the name of the subject in which we study the interaction of radiation with matter having prescribed emission and absorption coefficients.

Let us first consider how we can provide the mathematical description of radiation at a given point in space.

4.4.1 Radiation Field

You know that it is particularly easy to give a mathematical description of blackbody radiation, which is homogeneous and isotropic inside a container. Specifying the energy density u_ν associated with the frequency ν, which is given by Planck's law (Eq. 4.9), more or less provides us complete information about blackbody radiation. In general, however, the radiation is not isotropic. When we have sunlight streaming into a room, we obviously have a non-isotropic situation involving the flow of radiation from a preferred direction.

We now define the radiation field for a non-isotropic situation. Let us consider a small area dA at a point in space (Fig. 4.4). Let $dE_\nu\, d\nu$ be the energy of radiation passing through this area in time dt from the solid angle $d\Omega$ centred at θ and lying in the frequency range ν, $\nu + d\nu$. The energy $dE_\nu d\nu$ is proportional to the area $dA \cos\theta$ projected perpendicular to the direction of radiation, time interval dt, solid angle $d\Omega$ and frequency range $d\nu$. Hence we can write

$$dE_\nu d\nu = I_\nu(r, t, \hat{\mathbf{n}})\cos\theta\, dA\, dt\, d\Omega\, d\nu \quad (4.10)$$

where $\hat{\mathbf{n}}$ is the unit vector indicating the direction from which the radiation is coming and $\hat{\mathbf{N}}$ is the unit vector normal to the area dA. The quantity $I_\nu(r, t, \hat{\mathbf{n}})$ is called the **specific intensity**. As you can see, it is a function of position r, time t and direction $\hat{\mathbf{n}}$.

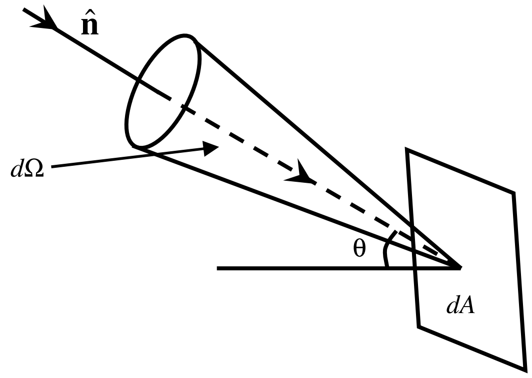

Fig.4.4: Illustration of specific intensity

Radiation field

If $I_\nu(r, t, \hat{\mathbf{n}})$ is specified for all directions at every point of a region at a time, then we say that the **radiation field** in that region is completely specified.

In this elementary treatment, we shall restrict ourselves only to radiation fields which are independent of time.

If we know the radiation field at a point in space we can calculate various quantities like **radiant flux**, **energy density** and **pressure of radiation**. For example, radiant flux is simply the total energy of radiation coming from all directions at a point per unit area per unit time. Hence, we simply have to divide Eq. (4.10) by $dA\ dt$ and integrate over all solid angles to get the flux. Thus, we can define the radiant flux associated with frequency ν, and the total radiant flux as follows:

Radiant flux in terms of specific intensity

The radiant flux associated with frequency ν is given by

$$F_\nu = \int I_\nu \cos\theta \, d\Omega \tag{4.11}$$

The **total radiant flux** is obtained by integrating over all frequencies

$$F = \int F_\nu d\nu \tag{4.12}$$

The pressure of the radiation field over a surface is given by the momentum exchanged per unit area per unit time, or **momentum flux**, perpendicular to that surface. Let us obtain an expression for momentum flux.

You know from Unit 3 of PHE-11 that the momentum associated with a photon of energy dE_ν is dE_ν / c.

Its component normal to the surface dA is $dE_\nu \cos\theta / c$. On dividing this by $dA\ dt$, we get the momentum flux associated with dE_ν ;

$$\text{Momentum flux} = \frac{dE_\nu \cos\theta}{c\, dA\, dt}$$

Using Eq. (4.10), we get the expression for momentum flux in terms of specific intensity:

$$\frac{dE_\nu \cos\theta}{c} \frac{1}{dAdt} = \frac{I_\nu}{c} \cos^2\theta\, d\Omega \qquad (4.13)$$

The radiation pressure p_ν is obtained by integrating the momentum flux over all directions.

Radiation pressure

$$p_\nu = \frac{1}{c} \int I_\nu \cos^2\theta\, d\Omega. \qquad (4.14)$$

If the radiation field is isotropic, i.e., it is independent of θ and ϕ, then

$d\Omega = \sin\theta\, d\theta\, d\varphi$ and $\int d\Omega = 4\pi$. Hence, we get,

$$p_\nu = \frac{I_\nu}{c} \int \cos^2\theta d\Omega = \frac{4\pi}{3} \frac{I_\nu}{c} \qquad (4.15)$$

We now apply these results to calculate energy density and radiation pressure.

SAQ 5

Spend 5 min.

Perform the integration in Eq. (4.15) and verify the result.

Example 3: Calculating energy density, specific intensity and radiation pressure

Calculate the energy density u_ν of a radiation field at a point and use that expression of energy density to write down the specific intensity of a blackbody radiation. Show that the pressure due to isotropic radiation is given by 1/3 of the energy density.

Solution

Let us consider energy dE_ν of radiation associated with frequency ν as given by Eq. (4.10). This energy passes through area dA in time dt in the direction $\hat{\mathbf{n}}$. Since the radiation traverses a distance cdt in time dt, we expect this radiation dE_ν to fill up a cylinder with base dA and length cdt in the direction $\hat{\mathbf{n}}$ during this time. Now the volume of this cylinder is cosθ $dAcdt$. Therefore, from Eq. (4.10), the energy density of this radiation in the solid angle $d\Omega$ is

$$\frac{dE_\nu}{\cos\theta\, dAcdt} = \frac{I_\nu}{c} d\Omega$$

To get the total energy density of radiation at a point associated with frequency ν, we have to integrate over all directions, so that

$$u_\nu = \int \frac{I_\nu}{c} d\Omega$$

For isotropic radiation $u_\nu = \dfrac{4\pi I_\nu}{c}$

Since blackbody radiation is isotropic, the specific intensity of blackbody radiation usually denoted by $B_\nu\,(T)$ should be independent of direction. Hence the energy density of blackbody radiation is simply given by

$$u_\nu = \int \frac{B_\nu(T)}{c} d\Omega$$

Since $B_\nu(T)$ is independent of direction, integration over all solid angles gives 4π.

$$\therefore \quad u_\nu = \frac{4\pi B_\nu(T)}{c}$$

Therefore, making use of Eq. (4.9), we now get the specific intensity of blackbody radiation as

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

Using Eq. (4.15), we get the radiation pressure for isotropic radiation as

$$p_\nu = \frac{1}{3} u_\nu$$

For black body radiation $p_\nu = \frac{4\pi}{3} \frac{B_\nu(T)}{c}$.

In astrophysics, we need to understand the interaction of matter and radiation to explain spectra of objects such as stars, interstellar gas clouds and galaxies. We now discuss the effect of matter on radiation field.

4.4.2 Radiative Transfer Equation

If matter is present, then in general the specific intensity of the radiation field keeps changing as we move along a ray path. Before we consider the effect of matter, first let us find out what happens to specific intensity in empty space as we move along a ray path.

See Fig. 4.5. Let dA_1 and dA_2 be two area elements separated by a distance R and perpendicular to a ray path. Let $I_{\nu 1}$ and $I_{\nu 2}$ be the specific intensities of radiation in the direction of the ray path at dA_1 and dA_2.

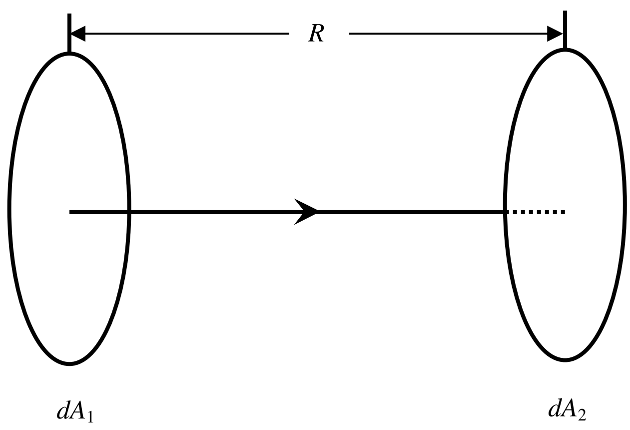

Fig.4.5: Two area elements perpendicular to a ray path

We want to determine the amount of radiation passing through both dA_1 and dA_2 in time dt in the frequency range ν, $\nu + d\nu$. If $d\Omega_2$ is the solid angle subtended by dA_1 at

dA_2, then according to Eq. (4.10), the radiation falling on dA_2 in time dt after passing through dA_1 is

$$I_{\nu 2} dA_2 dt d\Omega_2 d\nu$$

From considerations of symmetry, this should also be equal to

$$I_{\nu 1} dA_1 dt d\Omega_1 d\nu$$

where $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1. Equating these two expressions and noting that

$$d\Omega_1 = \frac{dA_2}{R^2}, \quad d\Omega_2 = \frac{dA_1}{R^2}$$

we get

$$I_{\nu 1} = I_{\nu 2} \tag{4.16}$$

In other words, in empty space the specific intensity along a ray path does not change. If s is the distance measured along the ray path, then we can write

$$\frac{dI_\nu}{ds} = 0 \tag{4.17}$$

in empty space.

At first sight, this may appear like a surprising result. We know that the intensity falls off as we move further and further away from a source of radiation. Can the specific intensity remain constant? The mystery is cleared when we keep in mind that the specific intensity due to a source is essentially its intensity divided by the solid angle it subtends, a quantity called the **surface brightness** of an object. This means that the specific intensity is a measure of the surface brightness. As we move further away from a source of radiation, both its intensity and angular size falls as $(\text{distance})^2$. Hence the surface brightness, which is the ratio of these two, does not change.

Let us now consider what happens if matter is present along the ray path. If matter emits, we expect that it will add to the specific intensity. This can be taken care of by adding an **emission coefficient** j_ν on the right hand side of Eq. (4.17). On the other hand, absorption by matter would lead to a diminution of specific intensity and the diminution rate must be proportional to the specific intensity itself. In other words, the stronger the beam, the more energy there is for absorption. Hence the absorption term on the right hand side of Eq. (4.17) should be negative and proportional to I_ν. Thus, we obtain the radiative transfer equation which gives the value of specific intensity in the presence of matter:

Radiative transfer equation

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \tag{4.18}$$

where α_ν is the **absorption coefficient**

The radiative transfer equation provides the basis for our understanding of interaction between radiation and matter.

It is fairly trivial to solve this equation if either the emission coefficient or the absorption coefficient is zero. Let us consider the case of $j_\nu = 0$, i.e., matter is assumed to absorb only but not to emit. Then Eq. (4.18) becomes

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{4.19}$$

On integrating this equation over the ray path from s_0 to s, we get

$$I_\nu(s) = I_\nu(s_0)\exp\left[-\int_{s_0}^{s} \alpha_\nu(s')ds'\right] \tag{4.20}$$

We will discuss below more general solutions of the radiative transfer equation. These solutions will provide us answers to questions such as: Why is the radiation emitted from nebula usually in spectral lines? Why do we see absorption lines in stellar spectra?

4.4.3 Optical depth; Solution of Radiative Transfer Equation

To obtain a general solution of the radiative transfer equation, we need to define two quantities, namely, the **optical depth** and the **source function**. Let us first define the **optical depth** τ_ν through the following relation:

$$d\tau_\nu = \alpha_\nu ds \tag{4.21}$$

such that the optical depth along the ray path between s_0 and s becomes

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')ds' \tag{4.22}$$

If matter does not emit radiation, i.e., $j_\nu = 0$, it follows from Eqs. (4.20) to (4.22) that the specific intensity along the ray path falls as

$$I_\nu(\tau_\nu) = I_\nu(0)\,e^{-\tau_\nu} \tag{4.23}$$

Based on the values of τ_ν we can define objects as optically thick or optically thin.

Optically thick and optically thin objects

If the optical depth $\tau_\nu >> 1$ along a ray path through an object, then the object is known as **optically thick**.

An object is known as **optically thin** if $\tau_\nu << 1$ for a ray path through it.

It follows from Eq. (4.23) that for an optically thick object $I_\nu(\tau_\nu) = 0$ and it extinguishes the light of a source behind it. What about an optically thin object? It does not decrease the light much. Hence the terms optically thick and optically thin roughly mean **opaque** and **transparent** at the frequency of electromagnetic radiation we are considering.

We now define the **source function**

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \tag{4.24}$$

Dividing the radiative transfer equation Eq. (4.18) by α_ν and using Eqs. (4.21) and (4.24), we get

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{4.25}$$

Multiplying this equation by e^{τ_ν}, we can write it as:

$$\frac{d}{d\tau_\nu}\left(I_\nu e^{\tau_\nu}\right)=S_\nu e^{\tau_\nu}$$

Integrating this equation from optical path 0 to τ_ν (i.e., from s_0 to s along the ray path), we get the **general solution** of the radiative transfer equation:

General solution of the radiative transfer equation

$$I_\nu(\tau_\nu)=I_\nu(0)e^{-\tau_\nu}+\int_0^{\tau_\nu} e^{-(\tau_\nu-\tau'_\nu)}S_\nu(\tau'_\nu)d\tau'_\nu \qquad (4.26)$$

If matter through which the radiation is passing has constant properties, then we can take S_ν to be constant and solve the integral in Eq. (4.26). This gives

$$I_\nu(\tau_\nu)=I_\nu(0)e^{-\tau_\nu}+S_\nu(1-e^{-\tau_\nu})$$

We are now interested in studying the emission and absorption properties of an object itself without a source behind it. Then we take $I_\nu(0) = 0$ and write

$$I_\nu(\tau_\nu)=S_\nu(1-e^{-\tau_\nu}) \qquad (4.27)$$

Let us consider the cases of optically thin and thick objects.

- **Optically thin object**

  For an optically thin object, $\tau_\nu << 1$, and $e^{-\tau_\nu}$ may be approximated to $1 - \tau_\nu$. Thus Eq. (4.27) becomes

  $$I_\nu(\tau_\nu)=S_\nu\tau_\nu$$

  For matter with constant properties, we take $\tau_\nu = \alpha_\nu L$, where L is the total length of the ray path. Making use of Eq. (4.24), we get the following result

  $$I_\nu = j_\nu L \qquad (4.28)$$

- **Optically thick object**

  If the object is optically thick, we can neglect $e^{-\tau_\nu}$ compared to 1 in Eq. (4.27). Then we get the result

  $$I_\nu = S_\nu \qquad (4.29)$$

  Let us put these results together for ready reference.

Specific Intensity of

Optically thin object: $I_\nu = j_\nu L$

Optically thick object: $I_\nu = S_\nu$

We have derived two tremendously important results in Eqs. (4.28) and (4.29). To understand their physical significance, we have to look at some thermodynamic considerations.

4.4.4 Local Thermodynamic Equilibrium

Suppose we have a box kept in thermodynamic equilibrium. If we make a small hole on its side, we know that the radiation coming out of the hole will be blackbody radiation. We have already derived the specific intensity of blackbody radiation as

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} \tag{4.30}$$

The specific intensity of radiation coming out of the hole is simply

$$I_\nu = B_\nu(T) \tag{4.31}$$

We now keep an optically thick object behind the hole inside the box as shown in Fig. 4.6. If this object is in thermodynamic equilibrium with the surroundings, then it will not disturb the environment and the radiation coming out of the hole will still be blackbody radiation, with specific intensity given by Eq. (4.31). On the other hand, we have seen in Eq. (4.29) that the radiation coming out of an optically thick object has the specific intensity equal to the source function. From Eq. (4.29) and (4.31), we conclude

$$S_\nu = B_\nu(T) \tag{4.32}$$

when matter is in thermodynamic equilibrium.

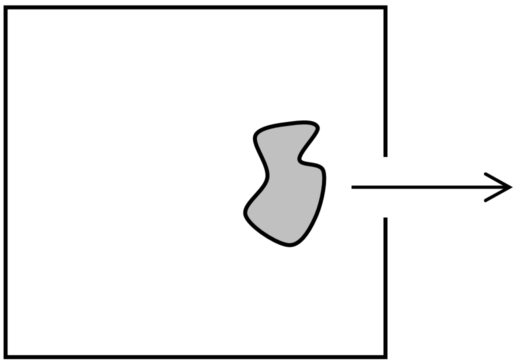

Fig.4.6: Blackbody radiation coming out of a box with an optically thick obstacle placed behind the hole inside the box

On using Eq. (4.24), we finally obtain the famous result known as Kirchoff's law.

Kirchoff's law

$$j_\nu = \alpha_\nu B_\nu(T) \tag{4.33}$$

Let us now stop and try to understand what we have derived. Very often matter tends to emit and absorb more at specific frequencies corresponding to spectral lines. Hence both j_ν and α_ν are expected to have peaks at spectral lines. But, according to Eq. (4.33), the ratio of these coefficients should be the smooth blackbody function $B_\nu(T)$.

We now look at the results of Eq. (4.28) and (4.29).

Eq. (4.28) tells us that the radiation from an optically thin source is essentially determined by its emission coefficient. Since the emission coefficient is expected to have peaks at spectral lines, we find that the emission from an optically thin system like a hot transparent gas is mainly in spectral lines.

On the other hand, from Eq. (4.29), the specific intensity of radiation from an optically thick source is its source function. This has been shown to be equal to the blackbody function $B_\nu(T)$ in Eq. (4.31). Hence, we expect an optically thick object like a hot piece of iron to emit roughly like a blackbody.

The nature of radiation from an astrophysical source crucially depends on whether the source is optically thin or optically thick. Emission from a tenuous nebula is usually in spectral lines. On the other hand, a star emits almost like a blackbody.

Why is the radiation from a star not exactly blackbody radiation? Why do we see absorption lines? Recall that we have derived Eq. (4.29) by assuming the source to have constant properties. This is certainly not true for a star. As we go down from the star's surface, temperature keeps increasing. Hence Eq. (4.29) should be only approximately true. It is the temperature gradient near the star's surface which gives rise to the absorption lines.

By assuming thermodynamic equilibrium, we have derived the tremendously important result Eq. (4.32) that the source function should be equal to the blackbody function $B_\nu(T)$. In a realistic situation, we rarely have strict thermodynamic equilibrium. The temperature inside a star is not constant, but varies with its radius. In such a situation, will Eq. (4.32) hold?

We have already mentioned in section 4.3 that the Maxwellian velocity distribution, the Boltzmann law and the Saha equation hold if the system is in thermodynamic equilibrium. This generally means that the temperature does not vary much over it. For Planck's law also to hold, the radiation has to interact with matter efficiently.

We note from the radiative transfer equation Eq. (4.18) that α_ν has the dimension of inverse length. Its inverse α_ν^{-1} gives the distance over which a significant part of a beam of radiation would get absorbed by matter. Often this distance α_ν^{-1} is referred to as the **mean free path** of photons, since this is the typical distance a photon is expected to traverse freely before interacting with an atom.

The smaller the value of α_ν^{-1}, the more efficient is the interaction between matter and radiation. If α_ν^{-1} is sufficiently small such that the temperature can be taken as constant over such distances, then we expect Planck's law of blackbody radiation to hold. In other words, if both α_ν^{-1} and the mean free path of particles are small compared to the length over which the temperature varies appreciably, then all the important laws of thermodynamic equilibrium hold.

Such a situation is known as *Local Thermodynamic Equilibrium*, which is abbreviated as LTE. Inside a star, we expect LTE to be a very good approximation and we can assume Eq. (4.32) to hold when we solve radiative transfer equation inside the star. In the outer atmosphere of a star, LTE may fail and it often becomes necessary to consider departures from LTE when studying the transfer of radiation there.

In this unit, we have discussed some basic principles of physics applicable to astrophysical systems. We now summarise the contents of this unit.

4.5 SUMMARY

- The **virial theorem**, states that the gravitational potential energy E_G and kinetic energy E_K of a system are of the same order

$$2E_K + E_G = 0$$

- Newton's theory of gravitation is generally adequate. However, if the quantity $f = \frac{2GM}{c^2 r}$ becomes of order unity in a system, then the general theory of relativity has to be used instead of Newton's theory.

- For astrophysical systems in thermodynamic equilibrium, the following results hold:

  Maxwellian velocity distribution

  $$dN_v = 4\pi N \left(\frac{m}{2\pi k_B T} \right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T} \right) dv$$

  Boltzmann's law

  $$\frac{N_e}{N_0} = \exp\left(-\frac{E}{k_B T} \right)$$

  Saha's equation

  $$\frac{n_{\text{II}}}{n_{\text{I}}} p_e = \left(\frac{2\pi m_e}{h^2} \right)^{3/2} (k_B T)^{5/2} \exp\left[-\frac{I}{k_B T} \right]$$

  Planck's law of black body radiation

  $$u_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T} \right) - 1}$$

- The **energy of radiation** lying in the frequency range ν and $\nu + d\nu$ passing through an area $dA \cos\theta$ in time dt from the solid angle $d\Omega$ centred at θ is given by

  $$dE_\nu d\nu = I_\nu (r, t, \hat{\mathbf{n}}) \cos\theta \, dA \, dt \, d\Omega \, d\nu$$

  where I_ν is called the **specific intensity**.

- The **radiant flux** of a time independent radiation field is defined in terms of specific intensity I_ν as

  $$F_\nu = \int I_\nu \cos\theta d\Omega$$

- The **radiation pressure** is given by

  $$p_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, d\Omega$$

- The interaction of matter with radiation is given by the **radiative transfer equation**

  $$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

  The **general solution** of this equation is given as

  $$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{t_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

- The **optical depth** of an object along a ray path between points s_0 and s is given by

$$\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')ds'$$

- **Kirchoff's law** for a system in thermodynamic equilibrium is given as

$$j_\nu = \alpha_\nu B_\nu(T)$$

4.6 TERMINAL QUESTIONS

Spend 30 *min.*

1. Suppose the Sun contracts at a uniform rate to half its present size in 10^7 yr. Suppose that all its energy is radiated from its surface. Calculate the luminosity of the Sun during the contracting phase.

2. Determine the size to which the Earth must shrink so that the use of Einstein's theory of gravitation becomes necessary.

3. Calculate the optical depth at which the specific intensity reduces to one-hundredth of its original value in a system in which no emission of radiation is taking place. Perform your calculation at a given frequency. Would the system be optically thick or thin?

4.7 SOLUTIONS AND ANSWERS

Self Assessment Questions (SAQs)

1. We need to equate the total kinetic energy of the globular star cluster with its gravitational potential energy. If the cluster has N stars with masses of individual stars of order m, and velocities of order v, the total kinetic of the star cluster is about

$$Nmv^2.$$

The total gravitational potential energy is

$$\frac{G(Nm)^2}{R},$$

On equating these two, we have

$$v^2 \approx \frac{GNm}{R},$$

from which

$$Nm \approx \frac{v^2 R}{G} \approx \frac{\left(10^6\right)^2 \times 10^{20}}{10^{-7}} \approx 10^{39}\,\text{g}$$

This is the mass of the cluster. Taking the masses of stars to be of order 10^{33}g (~ mass of the Sun), the cluster has about million stars in it.

2. $f = \dfrac{2GM}{c^2 r} = 1$

$$\Rightarrow \qquad r = \frac{2GM}{c^2} = \frac{2\times 6.673\times 10^{-11}\times 1.99\times 10^{30}}{\left(3\times 10^{8}\right)^2}\,\text{m}$$

$$= \frac{2\times 6.673\times 1.99}{9}\times 10^{3}\,\text{m}$$

$$= 2951 \text{ m}$$

3. $f = \dfrac{8\pi}{3}\dfrac{Gr^2\rho}{c^2} = \dfrac{8\pi}{3}.\dfrac{6.673\times 10^{-8}\times 10^{28}\times 10^{28}\times 10^{-29}}{3\times 10^{10}\times 3\times 10^{10}}$

$$= \frac{8\pi\times 6.673}{27}\times 10^{-1}$$

~ 0.6, which is of the order 1.

4. $\log p_e \dfrac{n_{\text{II}}}{n_{\text{I}}} = 2.5\log T - \dfrac{5040}{T}.I - 0.48$

$$\log p_e + \log\frac{n_{\text{II}}}{n_{\text{I}}} = 2.5\log T - \frac{5040}{T}.I - 0.48$$

$$\log \frac{n_{\text{II}}}{n_{\text{I}}} = -2 + 2.5\log T - \frac{5040}{T}.I - 0.48$$

$$= -2.48 + 2.5\log\left(10^4\right) - \frac{5040}{T}\times 13.6$$

$$= 0.67$$

5. $p_\nu = \dfrac{I_\nu}{c}\displaystyle\int \cos^2\theta . 2\pi . \sin\theta\, d\theta$

$$= 2\pi\frac{I_\nu}{c}.\frac{-\cos^3\theta}{3}\Bigg|_0^\pi = \frac{4\pi}{3}\frac{I_\nu}{c}$$

Terminal Questions

1. Gravitational potential energy of a star of radius $r = \dfrac{GM^2}{r}$

For the Sun of radius $R_\Theta/2$, gravitational P. E. $= \dfrac{2GM^2}{R_\Theta}$

Present gravitational potential energy of the Sun = $\dfrac{GM^2}{R_\Theta}$

$\therefore$ Energy radiated from its surface $= \dfrac{2GM^2}{R_\Theta} - \dfrac{GM^2}{R_\Theta} = \dfrac{GM^2}{R_\Theta}$

(Remember that when the star contacts, gravitational energy is released)

$$\therefore \qquad L = \frac{GM^2}{R_\Theta T} \qquad (T \text{ is the time of contraction})$$

$$= \frac{GM^2}{R_\Theta .10^7 \times 365 \times 24 \times 3600}$$

$$= \frac{6.673 \times 10^{-8} \times \left(2 \times 10^{33}\right)^2}{7 \times 10^{10} \times 10^7 \times 3.65 \times 10^2 \times 2.4 \times 10 \times 3.6 \times 10^3}$$

$$= \frac{6.673 \times 4}{7 \times 3.65 \times 2.4 \times 3.6} 10^{35} = \frac{6.673 \times 4 \times 10^{35}}{7 \times 3.65 \times 2.4 \times 3.6} \cdot \frac{1}{4 \times 10^{33}}$$

(in terms of present solar luminosity)

$$= \frac{6.673}{7 \times 3.65 \times 2.4 \times 3.6} \times 10^2$$

= 3.02 (in terms of present solar luminosity)

2. $f \approx 1 = \frac{2GM_{Earth}}{c^2 r}$

$$\therefore \quad r = \frac{2GM_{Earth}}{c^2}$$

$$= \frac{2 \times 6.673 \times 10^{-11} \times 6 \times 10^{24}}{\left(3 \times 10^8\right)^2} \text{ m (SI Units)}$$

$$= \frac{2 \times 6.673 \times 6}{9} \times 10^{-3} \text{ m}$$

$= 8.9 \times 10^{-3}$ m

3. Eq. (4.23) $\Rightarrow \quad \frac{I_\nu}{I_\nu(0)} = e^{-\tau_\nu}$

$$\frac{1}{100} = e^{-\tau_\nu} \quad \Rightarrow \quad \tau_\nu = \log_e(100) = 4.61$$

Since $\tau_\nu >> 1$, the system is optically thick.